The Muffin Problem

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How it Began

A Recreational Math Conference  
(Gathering for Gardner)  
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I found a pamphlet:

The Julia Robinson Mathematics Festival:  
A Sample of Mathematical Puzzles  
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets \( \frac{5}{3} \) where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
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Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?
Can We Do Better?

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Is there a procedure with a larger smallest piece?

YES WE CAN!
### Five Muffins, Three People—Proc by Picture

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

*Is there a procedure with a larger smallest piece?*
The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

NO WE CAN’T!
There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

**Case 0:** Some muffin is uncut. Cut it \( \left(\frac{1}{2}, \frac{1}{2}\right) \) and give both \( \frac{1}{2} \)-sized pieces to whoever got the uncut muffin. (Note \( \frac{1}{2} > \frac{5}{12} \).) Reduces to other cases.

(*Henceforth:* All muffins are cut into \( \geq 2 \) pieces.)

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. Then \( N \leq \frac{1}{3} < \frac{5}{12} \).

(*Henceforth:* All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets \( \geq 4 \) pieces. He has some piece

\[
\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}
\]

Great to see \( \frac{5}{12} \)
General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide $m$ muffins among $s$ students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a Mixed Int Program (in paper).
Amazing Results! / Amazing Theorems!

1. \( f(43, 33) = \frac{91}{264} \).
2. \( f(52, 11) = \frac{83}{176} \).
3. \( f(35, 13) = \frac{64}{143} \).

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant!

Have General Theorems from which upper bounds follow. Have General Procedures from which lower bounds follow.
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
More than $\frac{1}{5}$ of the students can share the muffins.

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffins $\left[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}\right]$
2. Divide 1 muffin $\left[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}\right]$
3. Give 4 students $\left(\frac{5}{20}, \frac{7}{20}\right)$
4. Give 1 student $\left(\frac{6}{20}, \frac{6}{20}\right)$

Can we do better?

No
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
4. Give 1 students $[\frac{6}{20}, \frac{6}{20}, \frac{6}{20}, \frac{6}{20}]$

Can we do better?

NO
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
2. Divide 1 muffin \( \left[ \frac{6}{12}, \frac{6}{12} \right] \)
3. Give 2 students \( \left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right) \)
4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

Theorem: \( f(m, s) = m \cdot s \cdot f(s, m) \).
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\(f(3, 5)\) proc is \(f(5, 3)\) proc but swap Divide/Give and mult by 3/5.
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
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4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by \( \frac{3}{5} \).

**Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).
Floor-Ceiling Thm (FC Thm) Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq FC(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lceil 2m/s \rceil} \right\} \right\}.$$  

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

Someone gets $\leq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \lfloor 2m/s \rfloor = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lceil 2m/s \rceil}$. 
THREE Students

**CLEVERNESS, COMP PROGS** for the procedure.

**FC Theorem** for optimality.

\[ f(1, 3) = \frac{1}{3} \]
\[ f(3k, 3) = 1. \]
\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \ k \geq 1. \]
\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]

**Note:** A Mod 3 Pattern.

**Theorem:** For all \( m \geq 3 \), \( f(m, 3) = FC(m, 3) \).
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

\[
f(4k, 4) = 1 \text{ (easy)}
\]

\[
f(1, 4) = \frac{1}{4} \text{ (easy)}
\]

\[
f(4k + 1, 4) = \frac{4k-1}{8k}, \ k \geq 1.
\]

\[
f(4k + 2, 4) = \frac{1}{2}.
\]

\[
f(4k + 3, 4) = \frac{4k+1}{8k+4}.
\]

Note: A Mod 4 Pattern.

Theorem: For all \( m \geq 4 \), \( f(m, 4) = FC(m, 4) \).

FC-Conjecture: For all \( m, s \) with \( m \geq s \), \( f(m, s) = FC(m, s) \).
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ except $m=11$, $f(m, 5) = FC(m, 5)$.
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.  
2. $f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lfloor 22/5 \rfloor}, 1 - \frac{11}{5\lceil 22/5 \rceil}\}\} = \frac{11}{25}$.  

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff=} 0.0066666\ldots$$

If $f(5, 11) < \frac{11}{25}$ then FC-conjecture is false!
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.
2. $f(11, 5) \leq \max\left\{\frac{1}{3}, \min\left\{\frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor}\right\}\right\} = \frac{11}{25}.$

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

If $f(5, 11) < \frac{11}{25}$ then FC-conjecture is false!

WE SHOW: $f(11, 5) = \frac{13}{30}$
There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

*(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)*
\( f(11, 5) = \frac{13}{30}, \) Easy Case Based on Students

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[ N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}. \]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the pieces is

\[ \geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}. \]

Look at the muffin it came from to find a piece that is

\[ \leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}. \]

(\textbf{Negation of Cases 2 and 3:} Every student gets 4 or 5 pieces.)
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
s_4 + s_5 = 5
\]

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.
\( f(11, 5) = \frac{13}{30} \), Fun Cases

**Case 4.1:** is \( \leq \frac{1}{2} \). Then there is a piece

\[
\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.
\]

The other piece from the muffin is

\[
\leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see} \quad \frac{13}{30}.
\]

**Case 4.2:** All 4-shares are \( > \frac{1}{2} \). So there are \( 4s_4 = 12 \) 4-shares. There are \( \geq 12 \) pieces \( > \frac{1}{2} \). Can’t occur.
1. Every muffin cut into two pieces.
2. Find $L$ such that some students get either $L$ or $L + 1$ pieces.
3. Find how many students get $L (L + 1)$ pieces.
4. Find intervals that these pieces must be in.
5. Find how many pieces are in an interval
6. Get a contradiction out of this.

Note: Can turn Interval Theorem into a function $INT$ such that $f(m, s) \leq INT(m, s)$. 
FC CONJECTURE STILL SORT OF TRUE

**FC Conj:** For all \( m \geq s \), \( f(m, s) = FC(m, s) \). FALSE

**Theorem:** For fixed \( s \), for \( m \geq \frac{s^3 + 2s^2 + s}{2} \) \( f(m, s) = FC(m, s) \).

**Statistics:** For \( 3 \leq s \leq 50 \), \( s + 1 \leq m \leq 59 \):

\[
\begin{align*}
  f(m, s) &= FC(m, s) \text{ in 683 cases} \\
  f(m, s) &= INT(m, s) \text{ in 194 cases}
\end{align*}
\]

Still 108 cases left. Need new technique!
The Buddy-Match Method! (BM)

Can FC and INT do everything?
No.
They are very good when \( \frac{2m}{s} > 3 \) but NOT so good otherwise.
We do a concrete example of The Buddy-Match Method

\[ f(43, 39) \leq \frac{53}{156} \]

(We have matching lower bound also)

**Definition:** Assume we have a protocol where all students get 2 or 3 shares. If \( x \) is a 2-share then the other share that student has is the shares match. Note that \( M(x) = \frac{m}{s} - x \).

**Warning:** We will apply \( M \) to intervals. These intervals have to have only 2-shares in them! But they will!
Theorem $f(43, 39) \leq \frac{53}{156}$ (≥ also known).
Assume there is an (43, 39)-procedure with smallest piece $> \frac{53}{156}$.
Can assume all muffins cut in 2 pieces, all students get $\geq 2$ shares.

**Case 1:** A student gets $\geq 4$ shares. Some share $\leq \frac{43}{39 \times 4} < \frac{53}{156}$.

**Case 2:** A student gets $\leq 1$ shares. Can’t occur.

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86.
How Many Students get Two Shares? Three Shares?

Let $s_2$ ($s_3$) be the number of 2-students (3-students).

$$2s_2 + 3s_3 \quad = \quad 86$$
$$s_2 + s_3 \quad = \quad 39$$

Get $s_2 = 31$ and $s_3 = 8$

**Case 3.1, 3.2, 3.3, 3.4:**

$(\exists)$ 3-share $\geq \frac{66}{156}$. Rm. Now 2-shares $\geq \frac{43}{39} - \frac{66}{156} = \frac{53}{78}$.

So some share $\leq \frac{53}{156}$.

By similar reasoning (Case 3.2, 3.3, 3.4) we have:

$$\left(\begin{array}{ccc}
\frac{53}{156} & 24 & 3\text{-shs} \\
\frac{66}{156} & 0 & \text{shs} \\
\frac{69}{156} & 62 & 2\text{-shs}
\end{array}\right)$$
The Buddy-Match Method

\[
\begin{pmatrix}
\frac{53}{156} & 24 \text{ 3-shs} \\
\frac{66}{156} & 0 \text{ shs} \\
\frac{69}{156} & 62 \text{ 2-shs} \\
\frac{103}{156}
\end{pmatrix}
\]

\[|(\frac{53}{156}, \frac{69}{156})| = 24\]

\[|B(\frac{53}{156}, \frac{69}{156})| = |\frac{87}{156}, \frac{103}{156}| = 24\]

\[|M(\frac{87}{156}, \frac{103}{156})| = |\frac{69}{156}, \frac{85}{156}| = 24\]

\[|(\frac{53}{156}, \frac{69}{156}) \cup (\frac{69}{156}, \frac{85}{156}) \cup (\frac{87}{156}, \frac{103}{156})| = 24 \times 3 = 72\]

\[|((\frac{85}{156}, \frac{87}{156})| = 86 - 72 = 14.\]
More Buddy-Match Method

\[ |(\frac{85}{156}, \frac{87}{156})| = 14. \text{ Buddy-Match yields } |(\frac{53}{156}, \frac{55}{156})| = 14 \]

\[ |[\frac{66}{156}, \frac{69}{156}]| = 0. \text{ Buddy-Match yields } |[\frac{55}{156}, \frac{58}{156}]| = 0. \]

The following picture captures what we know so far about 3-shares.

\[
\begin{pmatrix}
\frac{53}{156} & 14 & 0 \\
14 & \frac{55}{156} & 10 \\
0 & \frac{58}{156} & \frac{66}{156}
\end{pmatrix}
\]
Big Shares and Small Shares

\[
\begin{pmatrix}
53 & 14 \\
\frac{53}{156} & \frac{55}{156}
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{58}{156}
\end{pmatrix}
\begin{pmatrix}
10 \\
\frac{66}{156}
\end{pmatrix}
\]

- Shares in \(\left(\frac{53}{156}, \frac{55}{156}\right)\) are small shares;
- Shares in \(\left(\frac{58}{156}, \frac{66}{156}\right)\) are large shares;

**Notation** \(d_i\) is numb of students who have \(i\) small shares \((3 - i\) large shares).

\[d_0 = 0\] since \(3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39}\).

\[d_3 = 0\] since \(3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39}\).

SO there are NO \(d_0\)-students or \(d_3\)-students.
d$_1$ and d$_2$ Students Cause a Gap!

\[
\begin{pmatrix}
\frac{53}{156} & \frac{14}{156} \\
\frac{55}{156} & 0 \\
\frac{58}{156} & \frac{10}{156}
\end{pmatrix}
\]

\[d_1: \text{If a } d_1\text{-student has a large shares } \geq \frac{61}{156} \text{ then he will have}
\]
\[
\frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}.
\]

\textbf{Upshot:} Large shares of } d_1\text{-student are in } (\frac{58}{156}, \frac{61}{156}).

\[d_2: \text{If a } d_2\text{-student has a large shares } \leq \frac{62}{156} \text{ then he will have}
\]
\[
\frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}.
\]

\textbf{Upshot:} Large shares of a } d_2\text{-student are in } (\frac{62}{156}, \frac{66}{156}).

\textbf{Upshot Upshot:} There are NO shares in } [\frac{61}{156}, \frac{62}{156}]\]
Even More Buddy Match

The following picture captures what we know so far about 3-shares.

\[
\begin{pmatrix}
14 & 0 \\
\frac{53}{156} & \frac{55}{156}
\end{pmatrix}
\times
\begin{pmatrix}
0 & 14 \\
\frac{58}{156} & \frac{61}{156}
\end{pmatrix}
\times
\begin{pmatrix}
0 & y \\
\frac{62}{156} & \frac{66}{156}
\end{pmatrix}
\]

Use Buddy-Match to show that \( |(\frac{61}{156}, \frac{62}{156})| = |(\frac{62}{156}, \frac{63}{156})| \). So:

\[
\begin{pmatrix}
14 & 0 \\
\frac{53}{156} & \frac{55}{156}
\end{pmatrix}
\times
\begin{pmatrix}
0 & 14 \\
\frac{58}{156} & \frac{61}{156}
\end{pmatrix}
\times
\begin{pmatrix}
0 & 5 \\
\frac{63}{156} & \frac{66}{156}
\end{pmatrix}
\]

\[
x + y = 10.
\]

Use Buddy-Match to show that \( |(\frac{58}{156}, \frac{61}{156})| = |(\frac{63}{156}, \frac{66}{156})| \) so they are both 5.
Only the $d_2$-students use $\left(\frac{63}{156}, \frac{66}{156}\right)$. Every $d_2$ student uses one share from that interval:

$$d_2 = 5.$$ 

Each $d_i$ student uses $i$ shares from $\left(\frac{53}{156}, \frac{55}{156}\right)$:

$$1 \times d_1 + 2 \times d_2 = 14 : \text{ So } d_1 = 4$$

There are 8 3-students:

$$d_1 + d_2 = 8 : \text{ So } 5 + 4 = 8. \text{CONTRADICTION!}$$
The Essence of The Buddy-Match Method

1. Works when \( \lceil \frac{2m}{s} \rceil = 3 \): Just 2-shares and 3-shares.
2. \( 2m \) pieces, \( s_2 \) students get 2 shares, \( s_3 \) students get 3 shares.
3. Find a GAP
4. Using BM Sequence on 3-shares-interval find intervals that cover almost the entire interval. Missing an interval \((a, b)\).
5. Use BM on \((a, b)\) to get info on an initial interval of 3-shares.
6. Use BM on GAP to get GAPs within the 3-shares.
7. Set up linear equations relating intervals and types of students.
8. Show that system has no solution in \( \mathbb{N} \).

**Note:** Can turn BM technique into a function \( BM(m, s) \) such that \( f(m, s) \leq BM(m, s) \).
Statistics

For $3 \leq s \leq 60$, $s + 1 \leq m \leq 70$, $m, s$ rel prime:

\[ f(m, s) = FC(m, s) \text{ in } 927 \text{ cases. } \sim 68\% \]
\[ f(m, s) = INT(m, s) \text{ in } 268 \text{ cases. } \sim 20\% \]
\[ f(m, s) = BM(m, s) \text{ in } 85 \text{ cases. } \sim 6\% \]
\[ f(m, s) = ERIK(m, s) \text{ in } 80 \text{ cases. } \sim 6\% \]
All cases solved!
A Guess that Works. But Why?

1) We suspected there was a constant $X$ such that:
\[
(\forall k \geq 1) \left[ f(21k + 11, 21k + 4) \leq \frac{7k + X}{21k + 4} \right]
\]

2) We knew that $f(11, 4) = \frac{9}{20}$ so we conjectured $X = \frac{9}{5}$.

3) We prove the result with $X = \frac{9}{5}$ and $k \geq 1$ using BM. We prove matching lower bound for several $k$.

4) But the proof for $f(11, 4)$ ($k = 0$) cannot use BM and is totally unrelated to the proof for $k \geq 1$.

Note: This technique always worked!
Want to know $f(41, 19)$. Can’t use BM.  
$41 - 19 = 22$. So try to prove, diff $d$ is always Mod 3$d$ pattern. 
Need $X$: 

$$(\forall k \geq 1) \left[ f(66k + 41, 66k + 19) \leq \frac{22k + X}{66k + 19} \right]$$

Find $X$ using BM and linear algebra (have program for that). 
Get conj: $f(41, 19) = \frac{X}{19}$. 
**Note:** This seems to always work but have not been able to use to get new results yet.
We have a program that on input \((m, s)\):

1. We used FC, INT, BM to get upper bounds.
2. BM method is a theorem generator.
3. Use linear algebra to try to find a lower bound (a procedure).
Results

1. FC, INT, and BM upper bounds on $f(m, s)$
2. For fixed $s$, for $m \geq s^3$, $f(m, s) = FC(m, s)$.
3. For all $m \geq s$ $f(m, s) \geq \frac{1}{3}$.
4. For $1 \leq s \leq 7$ have proven formulas for $f(m, s)$. Mod $s$ pattern
5. For $s = 8, \ldots, 100$ conjectures for $f(m, s)$. $f(m, s)$ seems to be a mod $s$ pattern.
6. For $1 \leq d \leq 7$ have proven formulas for $f(s + d, s)$. Mod $3d$ pattern.
7. For all $d$ conjecture that our Theorem Generator gives $f(s + d, s)$.
8. Conjecture that for all $a, d$ there exists $X$ such that

$$(\forall k \geq 0)\left[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a}\right]$$
Open Problems—Complexity

Consider:
Given $m, s$ in binary, compute $f(m, s)$.

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.
2. Is it in NP? The procedure might be very large compared to the input.
3. Is it NP-complete or NP-hard?
4. The problem IS in FPT: $(\forall m \geq s^3)[f(m, s) = FC(m, s)]$. 
Conjectures That are Surely True

1. \( f(m, s) \) has mod \( s \) pattern with a few exceptions (known for large \( m \)).
2. \( f(s + d, s) \) has mod \( 3d \) pattern with no exceptions.
3. \( f(m, s) \) only depends on \( m/s \).
4. If \( f(m, s) = \frac{a}{b} \) then \( s \) divides \( b \).
Accomplishment I Am Most Proud of:

Convinced 4 High School students (Guang, Naveen, Naveen, Sunny)  
3 college student (Erik, Jacob, Daniel)  
1 professor (John D) that the most important field of Mathematics is Muffinry.
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