The Muffin Problem

Guangi Cui - Montgomery Blair HS
John Dickerson- University of MD
Naveen Durvasula - Montgomery Blair HS
William Gasarch - University of MD
Erik Metz - University of MD
Jacob Prinz-University of MD
Naveen Raman - Richard Montgomery HS
Daniel Smolyak- University of MD
Sung Hyun Yoo - Bergen County Academies (in NJ)
How it Began

A Recreational Math Conference
(Gathering for Gardner)
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I found a pamphlet:
The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets \( \frac{5}{3} \) where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

**Is there a procedure with a larger smallest piece?**

**VOTE**

- **YES**
- **NO**
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

YES WE CAN!

We use ! since we are excited that we can!
## Five Muffins, Three People—Proc by Picture

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<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
</tr>
<tr>
<td>Bob</td>
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</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$</td>
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</tbody>
</table>

Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T!

We use ! since we are excited to prove we can’t do better!
Five Muffins, Three People—Can't Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

(**Henceforth:** All muffins are cut into $\geq 2$ pieces.)

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.

(**Henceforth:** All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece \( \frac{5}{12} \).
2. NO Procedure for 5 muffins, 3 people, smallest piece > \( \frac{5}{12} \).

Amazing That Have Exact Result!
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $> \frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed! On Next Page!
Amazing Results!

1. Procedure for 43 muffins, 33 people, smallest piece $\frac{91}{264}$.
2. NO Procedure for 43 muffins, 33 people, smallest piece $> \frac{91}{264}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $> \frac{83}{176}$.

All done by hand, no use of a computer. Co-author Erik Metz is a muffin savant!
Amazing Results!

1. Procedure for 43 muffins, 33 people, smallest piece $\frac{91}{264}$.
2. NO Procedure for 43 muffins, 33 people, smallest piece $> \frac{91}{264}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $> \frac{64}{143}$.

All done by hand, no use of a computer. Co-author Erik Metz is a muffin savant!
Amazing Results!

1. Procedure for 43 muffins, 33 people, smallest piece \(\frac{91}{264}\).
2. NO Procedure for 43 muffins, 33 people, smallest piece \(>\ \frac{91}{264}\).

1. Procedure for 52 muffins, 11 people, smallest piece \(\frac{83}{176}\).
2. NO Procedure for 52 muffins, 11 people, smallest piece \(>\ \frac{83}{176}\).

1. Procedure for 35 muffins, 13 people, smallest piece \(\frac{64}{143}\).
2. NO Procedure for 35 muffins, 13 people, smallest piece \(>\ \frac{64}{143}\).

All done by hand, no use of a computer
Amazing Results!

1. Procedure for 43 muffins, 33 people, smallest piece $\frac{91}{264}$.
2. NO Procedure for 43 muffins, 33 people, smallest piece $\frac{91}{264}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.

All done by hand, no use of a computer

Co-author Erik Metz is a muffin savant!
General Problem

How can you divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

An $(m, s)$-procedure is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is optimal if it has the largest smallest piece of any procedure.

Let $f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a Mixed Int Program (in paper).
$f(3, 5) \geq ?$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
Think about it at your desk.
Clearly \( f(3, 5) \geq \frac{1}{5} \). Can we get \( f(3, 5) > \frac{1}{5} \)?

Think about it at your desk.

\( f(3, 5) \geq \frac{1}{4} \)

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

Think about it at your desk.

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $[\frac{5}{20}, \frac{7}{20}]$
4. Give 1 students $[\frac{6}{20}, \frac{6}{20}]$

Can we do better? Vote:
$f(3, 5) \geq ?$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
Think about it at your desk.
$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $\left(\frac{5}{20}, \frac{7}{20}\right)$
4. Give 1 students $\left(\frac{6}{20}, \frac{6}{20}\right)$

Can we do better? Vote:

YES
NO
UNKNOWN TO SCIENCE
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
Think about it at your desk.

1. Divide 2 muffin $\left[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}\right]$
2. Divide 1 muffin $\left[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}\right]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better? Vote:
- YES
- NO
- UNKNOWN TO SCIENCE

NO Proof on next slide.
There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

**Case 0:** Some student gets 1 piece, so size $\frac{3}{5}$. Cut that piece in half and give both $\frac{3}{10}$-sized pieces to that student. (Note $\frac{3}{10} > \frac{1}{4}$.) Reduces to other cases.  
(*Henceforth:* All students get $\geq 2$ pieces.)

**Case 1:** Some student gets $\geq 3$ pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5} < \frac{1}{4}$. (*Henceforth:* All students get 2 pieces.)

**Case 2:** All students get 2 pieces. 5 students, so 10 pieces. **Some muffin** gets cut into $\geq 4$ pieces. Some piece $\leq \frac{1}{4}$.
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
2. Divide 1 muffin \( \left[ \frac{6}{12}, \frac{6}{12} \right] \)
3. Give 2 students \( \left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right) \)
4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \(\left[\frac{5}{12}, \frac{7}{12}\right]\)
2. Divide 1 muffin \(\left[\frac{6}{12}, \frac{6}{12}\right]\)
3. Give 2 students \(\left(\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\right)\)
4. Give 1 students \(\left(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\right)\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \(\left[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}\right]\)
2. Divide 1 muffin \(\left[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}\right]\)
3. Give 4 students \(\left(\frac{5}{20}, \frac{7}{20}\right)\)
4. Give 1 students \(\left(\frac{6}{20}, \frac{6}{20}\right)\)

\(f(3, 5)\) proc is \(f(5, 3)\) proc but swap Divide/Give and mult by \(\frac{3}{5}\).
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
2. Divide 1 muffin \( \left[ \frac{6}{12}, \frac{6}{12} \right] \)
3. Give 2 students \( \left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right) \)
4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by \( \frac{3}{5} \).

**Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).
Floor-Ceiling Theorem (Generalize $f(5, 3) \leq \frac{5}{12}$)

$$f(m, s) \leq \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s \rceil}, 1 - \frac{m}{s\lceil 2m/s \rceil}\right\}\right\}.$$  

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

**Someone** gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s\lceil 2m/s \rceil}$.

**Someone** gets $\leq \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \lfloor \frac{1}{2m/s} \rfloor = \frac{m}{s\lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s\lceil 2m/s \rceil}$. 
CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \ k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \] (easy)

\[ f(1, 4) = \frac{1}{4} \] (easy)

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \; k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k - 1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k + 1}{8k + 4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO

YES but with some exceptions
FIVE Students, \( m = 1, \ldots, 11 \)

\[
f(1, 5) = \frac{1}{5} \quad \text{(easy or use } f(1, 5) = \frac{5}{1} f(5, 1).)\]

\[
f(2, 5) = \frac{1}{5} \quad \text{(easy or use } f(2, 5) = \frac{5}{2} f(5, 2).)\]

\[
f(3, 5) = \frac{1}{4} \quad \text{(use } f(3, 5) = \frac{3}{5} f(5, 3).)\]

\[
f(4, 5) = \frac{3}{10} \quad \text{(use } f(4, 5) = \frac{4}{5} f(5, 4).)\]

\[
f(5, 5) = 1 \quad \text{(Easy and fits pattern)}\]

\[
f(6, 5) = \frac{2}{5} \quad \text{(Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(7, 5) = \frac{1}{3} \quad \text{(Use Floor-Ceiling Thm, NOT pattern)}\]

\[
f(8, 5) = \frac{2}{5} \quad \text{(Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(9, 5) = \frac{2}{5} \quad \text{(Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(10, 5) = 1 \quad \text{(Easy and fits pattern)}\]

\[
f(11, 5) = \text{ (Will come back to this later)}\]
CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$
What About FIVE students, ELEVEN muffins?

$f(11, 5) \geq \frac{13}{30}$.

**Procedure:**

1. Divide 8 muffins into $(\frac{13}{30}, \frac{17}{30})$.
2. Divide 2 muffins into $(\frac{14}{30}, \frac{16}{30})$.
3. Divide 1 muffin into $(\frac{15}{30}, \frac{15}{30})$.
4. Give 2 students $[\frac{14}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{20}, \frac{13}{20}]$
5. Give 1 student $[\frac{17}{30}, \frac{17}{30}, \frac{16}{30}, \frac{16}{20}]$
6. Give 2 students $[\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{15}{30}]$. 
What About FIVE students, ELEVEN muffins?

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\} \leq 0.44. \]

So

\[ \frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666 \ldots \]
What About FIVE students, ELEVEN muffins?

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\} \leq 0.44. \]

So
\[ \frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots \]

**VOTE:**

1. \( f(11, 5) = \frac{13}{30} \): Needs NEW methods to bound \( f(m, s) \).
2. \( f(11, 5) = \frac{11}{25} \): Needs NEW better procedure.
3. \( f(11, 5) = \alpha \) where \( \frac{13}{30} < \alpha < \frac{11}{25} \). Needs both:
4. **UNKNOWN TO SCIENCE!**
What About FIVE students, ELEVEN muffins?

\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\} \leq 0.44. \]

So

\[ \frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666 \ldots \]

**VOTE:**

1. \( f(11, 5) = \frac{13}{30} \): Needs NEW methods to bound \( f(m, s) \).
2. \( f(11, 5) = \frac{11}{25} \): Needs NEW better procedure.
3. \( f(11, 5) = \alpha \) where \( \frac{13}{30} < \alpha < \frac{11}{25} \). Needs both:
4. **UNKNOWN TO SCIENCE!**

**KNOWN:** \( f(11, 5) = \frac{13}{30} \)

**HAPPY:** New opt tech more interesting than new proc.
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

*(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)*
\[ f(11, 5) = \frac{13}{30}, \text{ Easy Case Based on Students} \]

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the pieces is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

**(Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

\( s_4 \) is number of students who get 4 pieces

\( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22
\]

\[
s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 pieces.

\( s_5 = 2 \): There are 2 students who have 5 pieces.
\( f(11, 5) = \frac{13}{30}, \) Fun Cases

\[ \diamond \diamond \diamond \diamond \diamond \quad (\text{Sums to } 11/5) \]
\[ \diamond \diamond \diamond \diamond \diamond \quad (\text{Sums to } 11/5) \]

\[ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]
\[ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]
\[ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]

\[ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]

**Case 4.1:** One of (say)

\[ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]

is \( \leq \frac{1}{2} \). Then there is a piece

\[
\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.
\]

The other piece from the muffin is

\[
\leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.
\]
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4.2:** All

- ○ ○ ○ ○  (Sums to 11/5)
- ○ ○ ○ ○  (Sums to 11/5)
- ○ ○ ○ ○ ○  (Sums to 11/5)

are $> \frac{1}{2}$.
There are $\geq 12$ pieces $> \frac{1}{2}$. Can't occur.
The technique for $f(11, 5) \leq \frac{13}{30}$ has a generalization with a eight subcases. We do one concrete example:

$$f(24, 11) \leq \frac{19}{44}$$

(We have matching lower bound also)

**Definition:** Assume we have a protocol where all muffins are cut into two pieces. If $x$ is a piece then the other piece in the muffin it came from is its **buddy**. Note that $B(x) = 1 - x$. 
\[ f(24, 11) \leq \frac{19}{44} \]

**Theorem:** \( f(24, 11) \leq \frac{19}{44} \) (≥ also known)

Assume \((24, 11)\)-procedure with smallest piece \(> \frac{19}{44}\).
Can assume all muffin cut in two and all student gets \(\geq 2\) shares.
We show that there is a piece \(\leq \frac{19}{44}\).

**Case 1:** A student gets \(\geq 6\) shares. Some piece \(\leq \frac{24}{\frac{11}{6}} < \frac{19}{44}\).

**Case 2:** A student gets \(\leq 3\) shares. Some piece \(\geq \frac{24}{\frac{11}{3}} = \frac{8}{11}\).
Buddy of that piece \(\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}\).

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
How many students get 4? 5? Where are the Shares?

Let $s_4$ ($s_5$) be the number of 4-students (5-students).

\[
4s_4 + 5s_5 = 48 \\
s_4 + s_5 = 11 \quad \text{Get } s_4 = 7 \text{ and } s_5 = 4
\]

**Case 3.1:** $(\exists)$ 4-sh $\leq \frac{21}{44}$. Rm. Now: 3 shares $\geq \frac{24}{11} - \frac{21}{44}$. $(\exists)$ share

\[
\geq \left( \frac{24}{11} - \frac{21}{44} \right) = \frac{25}{44}.
\]

Buddy is

\[
\leq 1 - \frac{25}{44} = \frac{19}{44}.
\]

SO can assume all 4-shares are $> \frac{21}{44}$.

By similar reasoning:

**Case 3.2:** 4-shares in $\left( \frac{21}{44}, \frac{25}{44} \right)$, 5-shares in $\left( \frac{19}{44}, \frac{20}{44} \right)$.

\[
\begin{pmatrix}
19/44 & 20 \text{ 5-shs} & 0 \text{ shs} & 28 \text{ 4-shs} & 25/44
\end{pmatrix}
\]
Claim 1: There are no shares $x \in \left[ \frac{23}{44}, \frac{24}{44} \right]$.

If there was such a share then $B(x) \in \left[ \frac{20}{44}, \frac{21}{44} \right]$.

The following picture captures what we know so far.
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had \( \leq 2 \) L4 shares then he has

\[
< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.
\]

Contradiction: There are at least \( 3 \times s_4 = 3 \times 7 = 21 \) L4 shares. But there are only 20.
The Buddy-Match Method!

Can Floor-Ceiling and Interval-Method do everything? No. They are very good when \( \frac{2m}{s} > 3 \) but NOT so good otherwise. We do a concrete example of The Buddy-Match Method

\[
f(43, 39) \leq \frac{53}{156}
\]

(We have matching lower bound also)

**Definition:** Assume we have a protocol where all students get 2 or 3 shares. If \( x \) is a 2-share then the other share that student has is the shares match. Note that \( B(x) = \frac{m}{s} 1 - x \).

**Warning:** We will apply \( M \) to intervals. These intervals have to have only 2-shares in them! But they will!
Theorem $f(43, 39) \leq \frac{53}{156}$ (≥ also known).
Assume there is an (43, 39)-procedure with smallest piece > $\frac{53}{156}$. Can assume all muffins cut in 2 pieces, all students get ≥ 2 shares.

**Case 1:** A student gets ≥ 4 shares. Some share ≤ $\frac{43}{39 \times 4} < \frac{53}{156}$.

**Case 2:** A student gets ≤ 1 shares. Can’t occur.

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86.
How Many Students get Two Shares? Three Shares?

Let \( s_2 \) \((s_3)\) be the number of 2-students (3-students).

\[
2s_2 + 3s_3 = 86 \\
\quad s_2 + s_3 = 39 \text{ Get } s_2 = 31 \text{ and } s_3 = 8
\]

**Case 3.1, 3.2, 3.3, 3.4:**
(∃) 3-share \(\geq \frac{66}{156} \). Rm. Now 2-shares \(\geq \frac{43}{39} - \frac{66}{156} = \frac{53}{78} \).

So some share \(\leq \frac{53}{156} \).

By similar reasoning (Case 3.2, 3.3, 3.4) we have:

\[
\left( \begin{array}{c} 24 \text{ 3-shs} \\ \frac{53}{156} \end{array} \right) \left[ \begin{array}{c} 0 \text{ shs} \\ \frac{66}{156} \end{array} \right] \left( \begin{array}{c} 62 \text{ 2-shs} \\ \frac{69}{156} \end{array} \right) \quad \left( \begin{array}{c} 103 \\ \frac{103}{156} \end{array} \right)
\]
The Buddy-Match Method

\[
\begin{align*}
&\left( \begin{array}{c}
53 \\
156
\end{array} \right)[0 \begin{array}{c}
66 \\
156
\end{array}]\left( \begin{array}{c}
69 \\
156
\end{array} \right) \\
&\left( \begin{array}{c}
87 \\
156
\end{array} \right)\left( \begin{array}{c}
103 \\
156
\end{array} \right)
\end{align*}
\]

\[\left| \left( \begin{array}{c}
53 \\
156
\end{array} , \begin{array}{c}
69 \\
156
\end{array} \right) \right| = 24\]

\[\left| B\left( \begin{array}{c}
53 \\
156
\end{array} , \begin{array}{c}
69 \\
156
\end{array} \right) \right| = \left| \begin{array}{c}
87 \\
156
\end{array} , \begin{array}{c}
103 \\
156
\end{array} \right| = 24\]

\[\left| M\left( \begin{array}{c}
87 \\
156
\end{array} , \begin{array}{c}
103 \\
156
\end{array} \right) \right| = \left| \begin{array}{c}
69 \\
156
\end{array} , \begin{array}{c}
85 \\
156
\end{array} \right| = 24\]

\[\left| \left( \begin{array}{c}
53 \\
156
\end{array} , \begin{array}{c}
69 \\
156
\end{array} \right) \cup \left( \begin{array}{c}
69 \\
156
\end{array} , \begin{array}{c}
85 \\
156
\end{array} \right) \cup \left( \begin{array}{c}
87 \\
156
\end{array} , \begin{array}{c}
103 \\
156
\end{array} \right) \right| = 24 \times 3 = 72\]

\[\left| \left( \begin{array}{c}
85 \\
156
\end{array} , \begin{array}{c}
87 \\
156
\end{array} \right) \right| = 86 - 72 = 14.\]
More Buddy-Match Method

\[ \left| \left( \frac{85}{156}, \frac{87}{156} \right) \right| = 14. \text{ Buddy-Match yields } \left| \left( \frac{53}{156}, \frac{55}{156} \right) \right| = 14 \]

\[ \left| \left[ \frac{66}{156}, \frac{69}{156} \right] \right| = 0. \text{ Buddy-Match yields } \left| \left[ \frac{55}{156}, \frac{58}{156} \right] \right| = 0. \]

The following picture captures what we know so far about 3-shares.

\[
\begin{pmatrix}
\frac{53}{156} & 14 \\
\frac{55}{156} & 0
\end{pmatrix}
\begin{pmatrix}
\frac{58}{156} \\
\frac{66}{156}
\end{pmatrix}
\]
Big Shares and Small Shares

\[ \begin{pmatrix} 14 & 0 \\ \frac{53}{156} & \frac{55}{156} \end{pmatrix} \begin{pmatrix} 10 \\ \frac{58}{156} & \frac{66}{156} \end{pmatrix} \]

- Shares in \( \left( \frac{53}{156}, \frac{55}{156} \right) \) are small shares;
- Shares in \( \left( \frac{58}{156}, \frac{66}{156} \right) \) are large shares;

**Notation** \( d_i \) is numb of students who have \( i \) small shares (\( 3 - i \) large shares).

\[
d_0 = 0 \text{ since } 3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39}.
\]

\[
d_3 = 0 \text{ since } 3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39}.
\]

SO there are NO \( d_0 \)-students or \( d_3 \)-students.
$d_1$ and $d_2$ Students Cause a Gap!

\[
\begin{pmatrix}
\frac{53}{156} & 14 \\
\frac{55}{156} & 0 \\
\frac{58}{156} & 10
\end{pmatrix}
\begin{pmatrix}
\frac{156}{156} \\
\frac{156}{156} \\
\frac{156}{156}
\end{pmatrix}
\]

$d_1$: If a $d_1$-student has a large shares $\geq \frac{61}{156}$ then he will have

\[
\frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}.
\]

**Upshot:** Large shares of $d_1$-student are in \((\frac{58}{156}, \frac{61}{156})\).

$d_2$: If a $d_2$-student has a large shares $\leq \frac{62}{156}$ then he will have

\[
\frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}.
\]

**Upshot:** Large shares of a $d_2$-student are in \((\frac{62}{156}, \frac{66}{156})\).

**Upshot Upshot:** There are NO shares in $[\frac{61}{156}, \frac{62}{156}]$.
Even More Buddy Match

The following picture captures what we know so far about 3-shares.

\[
\begin{pmatrix}
\frac{53}{156} & 14 & 0 \\
\frac{55}{156} & 5 & 0 \\
\frac{58}{156} & x & 0 \\
\frac{61}{156} & y & 0
\end{pmatrix}
\]

\[x + y = 10.\]

Use Buddy-Match to show that \[|\left(\frac{58}{156}, \frac{61}{156}\right)| = |\left(\frac{63}{156}, \frac{66}{156}\right)|\] so they are both 5.

\[
\begin{pmatrix}
\frac{53}{156} & 14 & 0 \\
\frac{55}{156} & 5 & 0 \\
\frac{58}{156} & 5 & 0 \\
\frac{61}{156} & 5 & 0
\end{pmatrix}
\]
Only the $d_2$-students use $\left( \frac{63}{156}, \frac{66}{156} \right)$. Every $d_2$ student uses one share from that interval:

$$d_2 = 5.$$ 

Each $d_i$ student uses $i$ shares from $\left( \frac{53}{156}, \frac{55}{156} \right)$:

$$1 \times d_1 + 2 \times d_2 = 14 : \text{ So } d_1 = 4$$

There are 8 3-students:

$$d_1 + d_2 = 8 : \text{ So } 5 + 4 = 8. \text{CONTRADICTION!}$$
Summary of All of Our Results: Actual Numbers

1. For $1 \leq s \leq 50$ and $s + 1 \leq m \leq 60$ we have all $f(m, s)$ except

$$\frac{49}{114} \leq f(41, 19) < \frac{983}{2280}$$

$$f(48, 37) < \frac{103}{296}$$

$$f(50, 41) < \frac{59}{164}$$

2. A computer program that, on input $m, s$ uses our theorems to find $\alpha$ with $f(m, s) \leq \alpha$ and then tries to prove $f(m, s) \geq \alpha$ using linear algebra.

3. A Mixed Int Program for $f(m, s)$ which is too slow. Oh well.
1. For all $m \geq s$ $f(m, s) \geq \frac{1}{3}$.
2. Formulas for $f(m, s)$ for $1 \leq s \leq 7$. They are Mod $s$ patterns.
3. For $s = 8, \ldots, 100$ conjectures for $f(m, s)$. $f(m, s)$ seems to be a mod $s$ pattern.
4. Formulas for $f(s + d, s)$ for $1 \leq d \leq 7$ and a methodology to find more (Buddy Match). Seems to have a be a mod $3d$ pattern.
5. A computer program that, on input $m, s$ uses our theorems to find $\alpha$ with $f(m, s) \leq \alpha$ and then tries to prove $f(m, s) \geq \alpha$ using linear algebra.
Summary of our Results-Theorems

1. Floor Ceiling THEOREM, Interval THEOREM, Buddy Match METHODOLOGY to generate theorems.

2. For fixed $s$, for $m \geq \frac{s^3 + 2s^2 + s}{2}$ $f(m, s)$ matches the Floor-ceiling bound.

3. $f(m, s)$ always exists and is rational. Provable by compactness argument OR a large number of Linear Programs, OR one MIP. The last two proofs also give that $f(m, s)$ is computable.

4. Computer Generated Theorems like the following:
   For all $k \geq 0$, $f(21k + 11, 21k + 4) = \frac{7k + X}{21k + 4}$ where $X = \frac{9}{5}$.

5. Computer Generated Theorems like the following:
   If $1 \leq a \leq \frac{5d}{7}$, $a \neq \frac{d}{2}$, then $f(3dk + a + d, 3dk + a) \leq \frac{X}{156}$
   where $X \geq \max\{\frac{2a - d}{3}, \frac{7a - 2d}{8}, \frac{2a}{5}, \frac{a + d}{7}, \frac{4a - d}{5}, \frac{a + 2d}{10}\}$.
A Surely True But Really Strange Conjecture

We showed results like:

\[(\forall k \geq 0)[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a} \text{ where } X = BLAH].\]

\(X\) is independent of \(k\).

We conjectured the correct \(X\) by looking at \(k = 0\) case that we already knew.

We proved theorem for \(k \geq 1\) using Buddy-Match (did not apply to \(k = 0\) case).

Have theorem that works for \(k = 0\) with one proof, \(k = 1\) with another.
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\[(\forall k \geq 0)[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a} \text{ where } X = BLAH ].\]

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We conjectured the correct \(X\) by looking at \(k = 0\) case that we already knew.

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Have theorem that works for \(k = 0\) with one proof, \(k = 1\) with another.

**Conjecture** For all \(a, d\) with \(1 \leq a < d\), \(a, d\) relatively prime, there exists a constant \(X\) such that

\[(\forall k \geq 0)[f(3dk + a + d, 3dk + a) \leq \frac{X}{156} \text{ where } X = BLAH }.\]

(Exception when \(k = 0\) and \(a = 1\) when \(f(d + 1, 1) = 1\).)

Very Strange that so far this has ALWAYS held even though proof for \(k = 0\) and \(k > 1\) totally different!)
Consider:
Given $m, s$ in binary, compute $f(m, s)$.

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.
2. Is it in NP? The procedure might be very large compared to the input.
3. Is it NP-complete or NP-hard?
4. Given $m, s$ is there a bound on the denominators of the sizes of shares used?
Conjectures that are surely True

1. \( f(m, s) \) has mod \( s \) pattern (known for large \( m \)).
2. \( f(s + d, s) \) has mod 3\( d \) pattern.
3. \( f(m, s) \) only depends on \( m/s \).
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Convinced

▶ 4 High School students (Guang, Naveen, Naveen, Sunny)

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