The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
## Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

**Smallest Piece:** $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is \( \frac{1}{3} \).

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

YES WE CAN!

We use ! since we are excited that we can!
Five Muffins, Three People–Proc by Picture

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<td>RED</td>
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<td>$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$</td>
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Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T!

We use ! since we are excited to prove we can’t do better!
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

(Herefore: All muffins are cut into $\geq 2$ pieces.)

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.

(Herefore: All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $> \frac{5}{12}$.

Amazing That Have Exact Result!
1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $> \frac{5}{12}$.

**Amazing That Have Exact Result!**

Prepare To Be More Amazed! On Next Page!
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.

All done by hand, no use of a computer. Co-author Erik Metz is a muffin savant.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\gg \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\gg \frac{83}{176}$.

All done by hand, no use of a computer.

Co-author Erik Metz is a muffin savant.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $> \frac{64}{143}$.

All done by hand, no use of a computer

Co-author Erik Metz is a muffin savant
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\geq \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\geq \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\geq \frac{64}{143}$.

All done by hand, no use of a computer.
Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.

All done by hand, no use of a computer

Co-author Erik Metz is a *muffin savant*
General Problem

*How can you divide and distribute* $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

An $(m, s)$-procedure is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is *optimal* if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure.

We have shown $f(5, 3) = \frac{5}{12}$.

**Note:** $f(m, s) \geq \frac{1}{s}$: divide each muffin into $s$ pieces of size $\frac{1}{s}$ and give each student $m$ of them.
$f(3, 5) \geq ?$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$? Think about it at your desk.
Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$? Think about it at your desk.

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $[\frac{5}{20}, \frac{7}{20}]$
4. Give 1 students $[\frac{6}{20}, \frac{6}{20}]$

Can we do better? Vote: YES NO UNKNOWN TO SCIENCE NO Proof on next slide.
Clearly \( f(3, 5) \geq \frac{1}{5} \). Can we get \( f(3, 5) > \frac{1}{5} \)?

Think about it at your desk.

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

Can we do better? Vote:
f(3, 5) ≥?

Clearly \( f(3, 5) \geq \frac{1}{5} \). Can we get \( f(3, 5) > \frac{1}{5} \)?

Think about it at your desk.

\( f(3, 5) \geq \frac{1}{4} \)

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

Can we do better? Vote:

YES

NO

UNKNOWN TO SCIENCE
Clearly \( f(3, 5) \geq \frac{1}{5} \). Can we get \( f(3, 5) > \frac{1}{5} \)?

Think about it at your desk.

\( f(3, 5) \geq \frac{1}{4} \)

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

Can we do better? Vote:

YES

NO

UNKNOWN TO SCIENCE

NO  Proof on next slide.
There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

**Case 0:** Some student gets 1 piece, so size $\frac{3}{5}$. Cut that piece in half and give both $\frac{3}{10}$-sized pieces to that student. (Note $\frac{3}{10} > \frac{1}{4}$.) Reduces to other cases.

*(Henceforth: All students get $\geq 2$ pieces.)*

**Case 1:** Some student gets $\geq 3$ pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5} < \frac{1}{4}$.

*(Henceforth: All students get 2 pieces.)*

**Case 2:** All students get 2 pieces. 5 students, so 10 pieces. **Some muffin** gets cut into $\geq 4$ pieces. Some piece $\leq \frac{1}{4}$. 

$f(3, 5) \leq \frac{1}{4}$
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by 3/5.
3 People, 5 Muffins VS 5 People, 3 Muffins

\[ f(5, 3) \geq \frac{5}{12} \]

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 student \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffins \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 student \((\frac{6}{20}, \frac{6}{20})\)

\[ f(3, 5) \text{ proc is } f(5, 3) \text{ proc but swap Divide/Give and mult by } \frac{3}{5}. \]

**Theorem:** \[ f(m, s) = \frac{m}{s} f(s, m). \]
Floor-Ceiling Theorem (Generalize $f(5, 3) \leq \frac{5}{12}$)

$$f(m, s) \leq \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lceil 2m/s \rceil}\right\}\right\}.$$

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

**Someone** gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$. 

**Someone** gets $\leq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$. 

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lceil 2m/s \rceil}$. 
THREE Students

**CLEVERNESS, COMP PROGS** for the procedure.

**Floor-Ceiling Theorem** for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]
\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]
\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \ k \geq 1. \]
\[ f(4k + 2, 4) = \frac{1}{2}. \]
\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
YES but with some exceptions
FIVE Students, \( m = 1, \ldots, 11 \)

\[
f(1, 5) = \frac{1}{5} \text{ (easy or use } f(1, 5) = \frac{5}{1} f(5, 1).)\]

\[
f(2, 5) = \frac{1}{5} \text{ (easy or use } f(2, 5) = \frac{5}{2} f(5, 2).)\]

\[
f(3, 5) = \frac{1}{4} \text{ (use } f(3, 5) = \frac{3}{5} f(5, 3).)\]

\[
f(4, 5) = \frac{3}{10} \text{ (use } f(4, 5) = \frac{4}{5} f(5, 4).)\]

\[
f(5, 5) = 1 \text{ (Easy and fits pattern)}\]

\[
f(6, 5) = \frac{2}{5} \text{ (Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(7, 5) = \frac{1}{3} \text{ (Use Floor-Ceiling Thm, NOT pattern)}\]

\[
f(8, 5) = \frac{2}{5} \text{ (Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(9, 5) = \frac{2}{5} \text{ (Use Floor-Ceiling Thm, fits pattern)}\]

\[
f(10, 5) = 1 \text{ (Easy and fits pattern)}\]

\[
f(11, 5) = \text{ (Will come back to this later)}\]
CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$
What About FIVE students, ELEVEN muffins?

Procedure:

1. Divide 8 muffins into \((\frac{13}{30}, \frac{17}{30})\).
2. Divide 2 muffins into \((\frac{14}{30}, \frac{16}{30})\).
3. Divide 1 muffin into \((\frac{15}{30}, \frac{15}{30})\).
4. Give 2 students \([\frac{14}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{20}, \frac{13}{20}]\).
5. Give 1 student \([\frac{17}{30}, \frac{17}{30}, \frac{16}{30}, \frac{16}{20}]\).
6. Give 2 students \([\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{15}{30}]\).
What About FIVE students, ELEVEN muffins? Opt

Recall: **Floor-Ceiling Theorem:**

\[
f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \left\lfloor 2m/s \right\rfloor}, 1 - \frac{m}{s \left\lceil 2m/s \right\rceil} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \left\lfloor 22/5 \right\rfloor}, 1 - \frac{11}{5 \left\lceil 22/5 \right\rceil} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \times 5}, 1 - \frac{11}{5 \times 4} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{25}, \frac{9}{20} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \frac{11}{25} \right\} = \frac{11}{25} = 0.44.
\]
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** \( \frac{13}{30} \sim 0.43333 \leq f(11, 5) \)
- By **Floor-Ceiling** \( f(11, 5) \leq \frac{11}{25} \sim .44 \)

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666\ldots
\]
Where Are We On FIVE students, ELEVEN muffins?

- **By Procedure** \( \frac{13}{30} \sim 0.43333 \leq f(11, 5) \)
- **By Floor-Ceiling** \( f(11, 5) \leq \frac{11}{25} \sim .44 \)

So

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666 \ldots
\]

**Darling:** 0.0066666 close enough ?
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \sim 0.43333 \leq f(11, 5)$
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.0066666\ldots$$

**Darling:** 0.0066666 close enough?

**VOTE:**

1. $f(11, 5) = \frac{13}{30}$: Needs NEW technique to show limits on procedures.
2. $f(11, 5) = \frac{11}{25}$: Needs NEW better procedure.
3. $f(11, 5) = \alpha$ where $\frac{13}{30} < \alpha < \frac{11}{25}$. Needs both:
4. **UNKNOWN TO SCIENCE!**
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \sim 0.43333 \leq f(11, 5)$
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25} \sim .44$

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}, \text{ Diff} = 0.0066666\ldots$$

**Darling:** 0.0066666 close enough?

**VOTE:**

1. $f(11, 5) = \frac{13}{30}$: Needs NEW technique to show limits on procedures.
2. $f(11, 5) = \frac{11}{25}$: Needs NEW better procedure.
3. $f(11, 5) = \alpha$ where $\frac{13}{30} < \alpha < \frac{11}{25}$. Needs both:
4. **UNKNOWN TO SCIENCE!**

**KNOWN:** $f(11, 5) = \frac{13}{30}$

**HAPPY:** New opt tech more interesting than new proc.
There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it ($\frac{1}{2}$, $\frac{1}{2}$) and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)
\( f(11, 5) = \frac{13}{30}, \) Easy Case Based on Students

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the pieces is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)
Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

$$4s_4 + 5s_5 = 22$$
$$s_4 + s_5 = 5$$

$s_4 = 3$: There are 3 students who have 4 pieces.
$s_5 = 2$: There are 2 students who have 5 pieces.
$f(11, 5) = \frac{13}{30}$, Fun Cases

\[\begin{array}{cccccc}
\Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \text{(Sums to 11/5)} \\
\Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \Diamond & \text{(Sums to 11/5)}
\end{array}\]

\[\begin{array}{cccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \text{(Sums to 11/5)} \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \text{(Sums to 11/5)}
\end{array}\]

\[\begin{array}{cccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \text{(Sums to 11/5)} \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \text{(Sums to 11/5)}
\end{array}\]

**Case 4.1:** One of (say)

\[\begin{array}{cccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \text{(Sums to 11/5)}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \text{(Sums to 11/5)}
\end{array}\]

is $\leq \frac{1}{2}$. Then there is a piece

$$\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.$$ 

The other piece from the muffin is

$$\leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.$$
Case 4.2: All

- O O O O (Sums to 11/5)
- O O O O (Sums to 11/5)
- O O O O (Sums to 11/5)

are > \( \frac{1}{2} \).

There are \( \geq 12 \) pieces > \( \frac{1}{2} \). Can't occur.
The Techniques Generalizes!

**Good News!**
The technique used to get $f(11, 5) \leq \frac{13}{30}$ lead to a theorem that apply to other cases! We call it **The Interval Theorem**

**Bad News!**
*Interval Theorem* is hard to state, so you don’t *get* to see it.

**Good News!**
*Interval Theorem* is hard to state, so you don’t *have* to see it.
Notation

$FC(m, s)$ is the upper bound provided by Floor-Ceiling Thm.

$IN(m, s)$ is the upper bound provided by INterval Thm.

$SP(s + 1, s) = f(s + 1, s)$. We have a theorem that tells us this exactly.
How Good Is the FC Bound? Mod Pattern?

1. For all $s$ for all $m \geq \frac{s^3 + 2s^2 + s}{2}$, $f(m, s) = FC(m, s)$. (Empirical evidence $O(s^2)$).

2. For all $s$ there is a mod-$s$-formula $FORM(m, s)$ such that for all $m \geq \frac{s^2 + s}{4}$, $f(m, s) = FORM(m, s)$.

3. Hence: For all $s$ there is a mod-$s$-formula $FORM(m, s)$ such that for all $m \geq \frac{s^3 + 2s^2 + s}{2}$, $f(m, s) = FORM(m, s)$.

4. For $1 \leq s \leq 6$ we have the $FORM(m, s)$.

5. For $7 \leq s \leq 60$ have conjectures for $FORM(m, s)$ that are surely true.
The Exceptions

For all $s$ there is a mod-$s$-formula $\text{FORM}(m, s)$ such that for all $m \geq \frac{s^2 + s}{4}$, $f(m, s) = \text{FORM}(m, s)$.

What happens when $\text{FORM}(m, s) \neq f(m, s)$.

1. $f(s + 1, s)$. Have Sep theorem for that case, known exactly.
2. $f(m, s) = \frac{1}{3}$.
3. $f(m, s)$ used Interval Theorem.

So far these are the only exceptions.
Does $f(m, s)$ Exist? Rational? Debatable?

**Plausible:**

1. There is a protocol showing $f(m, s) \geq \frac{1}{5}$
2. There is a protocol showing $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2}$
3. There is a protocol showing $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}$
4. :

But NO protocol shows $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{4}$.
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**Plausible:** $f(m, s) = \frac{1}{\pi}$ (so $\pi$ is key to muffins!)
Does $f(m, s)$ Exist? Rational? Debatable?

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But NO protocol shows $f(m, s) \geq \frac{1}{5} + \frac{1}{5^2} + \cdots = \frac{1}{4}$.

**Plausible:** $f(m, s) = \frac{1}{\pi}$ (so $\pi$ is key to muffins!)

**Plausible:** $f(m, s)$ is not computable.
Theorem

1. There is a mixed integer program with $O(ms)$ binary variables, $O(ms)$ real variables, $O(ms)$ constraints, and all coefficients integers of absolute value $\leq \max\{m, s\}$ such that, from the solution, one can extract $f(m, s)$ and a protocol that achieves this bound. This MIP can easily be obtained given $m, s$.

2. $f(m, s)$ is always rational. This follows from part 1.

3. The problem of, given $m, s$, determine $f(m, s)$, is decidable. This follows from part 1.
Not Just Theoretical

**Good News:** $f(m, s)$ exists, is rational and computable!
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**Bad News:** There is no more bad news which breaks the symmetry of good/bad/good/bad.

**Good News:** We HAVE coded it up and we HAVE gotten some results this way.
The Synergy Between Fields

One often hears:

**Pure Math done without an application in mind often ends up being Applied!**

(Number theory and Cryptography is a great example.)
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(MIP and Muffins is a ‘**great**’ example.)

**Pure Math, Applied Math, Computer Science, Physics**, all play off each other! None of the four has moral superiority!
How Research Works

1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (11,5) and (35,13)).
4. Lather, Rinse, Repeat.
Conjectures

**Conjecture:** The following program computes \( f(m, s) \) for \( m > s \).

- If \( d = \gcd(m, s) \neq 1 \) then call \( f(m/d, s/d) \).
- If \( m = s + 1 \) output \( SP(s + 1, s) \).
- If \( s = 1 \) then output 1.
- Otherwise output the \( \min \) of \( FC(m, s) \) and \( INT(m, s) \)

Empirically true for \( 1 \leq s \leq 15 \), \( 1 \leq m \leq 100 \).

**If True:**

1. \( f(m, s) \) can be computed with a constant number of arith operations on numbers \( \leq O(s + m) \).
2. \( f(m, s) \) can be computed in time \( O(M(s + m)) \), where \( M \) is speed of multiplication.
3. \( f(m, s) \) is in \( P \).
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