The Muffin Problem

Guangi Cui - Montgomery Blair HS (in MD)
Naveen Durvasula - Montgomery Blair HS (in MD)
William Gasarch - University of MD
Naveen Raman - Richard Montgomery HS (in MD)
Sung Hyun Yoo - Bergen County Academies (in NJ)
Cake Cutting

1. **Proportional Cake Cutting**: \( n \) people divide and distribute a cake so that everyone has \( \frac{1}{n} \) in their opinion. Exists \( O(n \log n) \) cuts discrete protocols. Optimal. **Crumbs!**

2. **Envy Free Cake Cutting**: \( n \) people divide and distribute a cake so that everyone has biggest (or tied) piece in their opinion. Exists \( O(n^n) \) (six \( n \)'s) cuts discrete protocols. No lower bounds known. **Crumbs!!!!** (Prior result had been unbounded protocol. This result was a surprise.)

3. **Cake Cutting** is a long studied problems. **Many** paper in Theory (Okay) and AI (What?).

4. **This Talk** is not about traditional cake cutting.
Our “Motivation”

1. Want to avoid crumbs.
2. All people will have uniform tastes. \( \alpha \) of a cake is of value \( \alpha \).
3. We use muffins rather than cakes.
4. **Honesty:** This is motivation after the fact.
Five Muffins, Three Students

At

**Gathering for Gardner Conference**

I found a pamphlet advertising

**The Julia Robinson Mathematics Festival**

which had this problem, proposed by Alan Frank:

*How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?*
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

**Is there a procedure with a larger smallest piece?**

**VOTE**
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is \( \frac{1}{3} \).

**Is there a procedure with a larger smallest piece?**

**VOTE**

- **YES**
- **NO**

**YES WE CAN!**

We use ! since we are excited that we can!
### Five Muffins, Three People–Proc by Picture

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**Smallest Piece:** $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T! We use ‘!’ since we are excited to prove we can’t do better!
Assumption We Can Make

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

We **ASSUME** each muffin cut into **at least 2** pieces: If not then cut that muffin \((\frac{1}{2}, \frac{1}{2})\).

**THIS TALK** ALL proofs will be about opt being $\leq 1/2$. We assume each muffin is cut into **at least 2** pieces.

**PIECES VS SHARES:** They are the same.
- **PIECE** is muffin-view,
- **SHARE** is student-view.
Muffin Principle

If a muffin is cut into $\geq u$ pieces then there is a piece $\leq \frac{1}{u}$.

Example: If a Muffin cut into 3 pieces:

some piece is $\leq \frac{1}{3}$. 
If a student gets $\geq u$ shares then there is a share $\leq \frac{m}{s} \times \frac{1}{u}$

Example: 5 muffins, 3 students. All student gets $\frac{5}{3}$.

If some student gets $\geq 4$ shares:

Then one of these pieces is $\leq \frac{5}{3} \times \frac{1}{4}$
Pieces Principle

If there are $P$ pieces then:
Some student gets $\geq \lceil \frac{P}{s} \rceil$
Some student gets $\leq \lfloor \frac{P}{s} \rfloor$

Example: 5 muffins, 3 people. If there are 10 pieces:

Some student gets $\geq \lceil \frac{10}{3} \rceil = 4$
Some student gets $\leq \lfloor \frac{10}{3} \rfloor = 3$
Five Muffins, Three People—Can’t Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.
(Negation: All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$
Great to see $\frac{5}{12}$
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $> \frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed!

We have many results like this!:

\[
\begin{align*}
f(47, 9) &= \frac{111}{234} \\
f(52, 11) &= \frac{83}{176} \\
f(35, 13) &= \frac{64}{143}
\end{align*}
\]
**General Problem**

*How can you divide and distribute* $m$ *muffins to* $s$ *students so that each student gets* $\frac{m}{s}$ *AND the MIN piece is MAXIMIZED?*

Let $m, s \in \mathbb{N}$.

An $(m, s)$-procedure is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is *optimal* if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure. ($f(m, s)$ exists. Compactness argument by *Douglas Ulrich*.)

We have shown $f(5, 3) = \frac{5}{12}$. 
**Terminology Issue**

Let $m, s \in \mathbb{N}$.

$m$ is the number of muffins.

$s$ is the number of students.

1. $f(m, s) \geq \alpha$ means that there is a procedure with smallest piece $\alpha$. We call this *A Procedure*.

2. $f(m, s) \leq \alpha$ means that there is NO procedure with smallest piece $> \alpha$. We call this *An Optimality Result* or *An Opt Result*.

**DO NOT** use terms *upper bound* and *lower bounds*:

1. Procedures are lower bounds, *opposite* of usual terminology.

2. Opt results are upper bounds, *opposite* of usual terminology.
Floor-Ceiling Theorem

\[ f(m, s) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}. \]

Proof:

Case 1: Some muffin is cut into \( \geq 3 \) pieces. Some piece \( \leq \frac{1}{3} \).

Case 2: Every muffin is cut into 2 pieces, so \( 2m \) pieces.

Someone gets \( \geq \left\lceil \frac{2m}{s} \right\rceil \) pieces. Some piece is \( \leq \frac{(m/s)}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil} \).

Someone gets \( \leq \left\lfloor \frac{2m}{s} \right\rfloor \) pieces. Some piece is \( \geq \frac{(m/s)}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor} \).

The other piece from that muffin is of size \( \leq 1 - \frac{m}{s \lfloor 2m/s \rfloor} \).
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \] (easy)
\[ f(1, 4) = \frac{1}{4} \] (easy)
\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]
\[ f(4k + 2, 4) = \frac{1}{2}. \]
\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO

NO! (excited because YES would be boring)
FIVE Students, \( m = 1, 2, 3, 4, 7, 11, 10k \)

\[
\begin{align*}
f(1, 5) &= \frac{1}{5} \text{ (easy)} \\
f(2, 5) &= \frac{1}{5} \text{ (easy)} \\
f(3, 5) &= \frac{1}{4} \text{ (Like } f(5, 3) = \frac{5}{12} \text{ but Muffins/Students reversed)} \\
f(4, 5) &= \frac{3}{10} \text{ (Will discuss briefly later)} \\
f(7, 5) &= \frac{1}{3} \text{ (use Floor-Ceiling Thm)} \\
f(11, 5) &= \text{ (Will come back to this later)} \\
f(10k, 5) &= 1 \text{ (Trivial)}
\end{align*}
\]
FIVE Students

Results on the next few slides:

**CLEVERNESS, COMP PROGS** for the procedure.

**Floor-Ceiling Theorem** for optimality.
FIVE Students $m = 10k + 1, 10k + 2, 10k + 3$

If $k$ not specified then $k \geq 0$.

$m = 10k + 1$:

\[ f(30k + 1, 5) = \frac{30k+1}{60k+5} \]
\[ f(30k + 11, 5) = \frac{30k+11}{60k+25} \quad (k \geq 1) \]
\[ f(30k + 21, 5) = \frac{10k+7}{20k+15} \]

$f(10k + 2, 5) = \frac{10k-2}{20k} \quad (k \geq 1)$

$f(10k + 3, 5) = \frac{10k+3}{20k+10} \quad (k \geq 1)$
FIVE Students $m = 10k + 4, 10k + 5, 10k + 6$

$$m = 10k + 4$$

$$f(30k + 4, 5) = \frac{30k+1}{60k+5}$$

$$f(30k + 14, 5) = \frac{30k+11}{60k+25}$$

$$f(30k + 24, 5) = \frac{10k+7}{20k+15}$$

$$f(10k + 5, 5) = 1$$

$$m = 10k + 6:$$

$$f(30k + 6, 5) = \frac{10k+2}{20k+5}$$

$$f(30k + 16, 5) = \frac{30k+16}{60k+35}$$

$$f(30k + 26, 5) = \frac{30k+26}{60k+55}$$
FIVE Students $m = 10k + 7, 10k + 8, 10k + 9$

$$f(10k + 7, 5) = \frac{10k + 3}{20k + 10}$$

$$f(10k + 8, 5) = \frac{5k + 4}{10k + 10}$$

$m = 10k + 9$

$$f(30k + 9, 5) = \frac{10k + 2}{20k + 5}$$

$$f(30k + 19, 5) = \frac{30k + 16}{60k + 35}$$

$$f(30k + 29, 5) = \frac{30k + 26}{60k + 55}$$
What About FIVE students, ELEVEN muffins?

Procedure:

Divide the Muffins in to Pieces:

1. Divide 6 muffins into $\left(\frac{13}{30}, \frac{17}{30}\right)$.
2. Divide 4 muffins into $\left(\frac{9}{20}, \frac{11}{20}\right)$.
3. Divide 1 muffin into $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Distribute the Shares to Students:

1. Give 2 students $\left[\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2}\right]$.
2. Give 2 students $\left[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20}\right]$.
3. Give 1 student $\left[\frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20}\right]$.

So

$$f(11, 5) \geq \frac{13}{30}$$
Recall: **Floor-Ceiling Theorem:**

\[
f(m, s) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{11}{5 \times 5}, 1 - \frac{11}{5 \times 4} \right\} \right\}.
\]

\[
f(11, 5) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{11}{25}, \frac{9}{20} \right\} \right\}.
\]

\[
f(11, 5) \leq \max\left\{ \frac{1}{3}, \frac{11}{25} \right\} = \frac{11}{25}.
\]
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \leq f(11, 5)$.
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}$$

Diff $= 0.006666\ldots$
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \leq f(11, 5)$.
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff}= 0.006666\ldots$$

**VOTE:**

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students 3 wrote in Harambe.)
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \leq f(11, 5)$.
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}$$

Diff$= 0.006666 \ldots$

**VOTE:**

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students 3 wrote in Harambe.)

**KNOWN:** $f(11, 5) = \frac{13}{30}$

**HAPPY:** New opt tech more interesting than new proc.
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

$N$ is smallest piece.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

*(Negation: All muffins cut into 2 pieces.)*
\( f(11, 5) = \frac{13}{30} \), Easy Case Based on Students

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the shares is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

(*Negation of Cases 2 and 3:* Every student gets 4 or 5 shares.)
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 shares
- $s_5$ is number of students who get 5 shares

\[
4s_4 + 5s_5 = 22 \\
\quad s_4 + s_5 = 5
\]

$s_4 = 3$: There are 3 students who have 4 shares.  
$s_5 = 2$: There are 2 students who have 5 shares.
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

\[ \diamond \text{ and } \circ \text{ are shares.} \]

\[ \diamond \diamond \diamond \diamond \diamond \quad \text{(Sums to 11/5)} \]
\[ \diamond \diamond \diamond \diamond \diamond \quad \text{(Sums to 11/5)} \]

\[ \circ \circ \circ \circ \circ \quad \text{(Sums to 11/5)} \]
\[ \circ \circ \circ \circ \circ \quad \text{(Sums to 11/5)} \]
\[ \circ \circ \circ \circ \circ \circ \quad \text{(Sums to 11/5)} \]

\[ \text{Case 3.1: One of (say)} \]

\[ \circ \circ \circ \circ \circ \quad \text{(Sums to 11/5)} \]

is \( \leq \frac{1}{2} \). Then there is a share

\[ \geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}. \]

The other piece from the muffin is

\[ \leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}. \]
\( f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \)

**Case 3.2: All**

- ○ ○ ○ ○ ○ \quad \text{(Sums to 11/5)}
- ○ ○ ○ ○ ○ \quad \text{(Sums to 11/5)}
- ○ ○ ○ ○ ○ \quad \text{(Sums to 11/5)}

are \( > \frac{1}{2} \).

There are \( \geq 12 \) shares \( > \frac{1}{2} \). Can’t occur.
Good News!
The technique used to get $f(11, 5) \leq \frac{13}{30}$ lead to a theorem that apply to other cases!

Bad News!
The theorem is hard to state, so you don’t get to see it.

Good News!
The theorem is hard to state, so you don’t have to see it.
What Else Have We Accomplished?

1. 13 theorems to help get us opt results.
2. $f(s, s - 1)$ and $f(s + 1, s)$ completely known.
3. A computer program that helps us get procedures.
4. For $1 \leq s \leq 15$, for all $m$, know $f(m, s)$.
5. Prove that $f(m, s)$ is always rational and the function is computable.
6. Convinced 4 High School students that the most important field of Mathematics is Muffinry.
Open Questions

1. For all $s$ there is a pattern for $f(m, s)$ that depends on $m \mod T$ where $s$ divides $T$.
2. $f(m, s) = \frac{a}{b}$ (lowest terms) where $s$ divides $b$. 
History and Hope

**History:**

1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (4,5) and (11,5)).
4. Lather, Rinse, Repeat.

**Hope:** A finite set of theorems that settle all cases.

**Likely?**

1. I think No, but

2. was surprised by the $n$-person $O(n^{n\cdot\cdot\cdot})$ cuts envy free cake cutting algorithm, so I could be surprised again!