The Muffin Problem

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Five Muffins, Three Students

At

A Recreational Math Conference
(Gathering for Gardner)

I found a pamphlet advertising
The Julia Robinson Mathematics Festival
which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets 5/3 where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
</tr>
</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

YES WE CAN!

We use ! since we are excited that we can!
Five Muffins, Three People–Proc by Picture

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<td>$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$</td>
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Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

**Is there a procedure with a larger smallest piece?**

**VOTE**

- **YES**
- **NO**
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

Is there a procedure with a larger smallest piece?

VOTE

- YES
- NO

NO WE CAN’T!

We use ! since we are excited to prove we can’t do better!
**Assumption We Can Make**

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

We **ASSUME** each muffin cut into at least 2 pieces: If not then cut that muffin $(\frac{1}{2}, \frac{1}{2})$.

**THIS TALK** ALL proofs will be about opt being $\leq 1/2$. We assume each muffin is cut into at least 2 pieces.

**PIECES VS SHARES:** They are the same.
- **PIECE** is muffin-view,
- **SHARE** is student-view.
Muffin Principle

If a muffin is cut into $\geq u$ pieces then there is a piece $\leq \frac{1}{u}$.

Example: If a Muffin cut into 3 pieces:

some piece is $\leq \frac{1}{3}$. 
Student Principle (not Principal)

If a student gets $\geq u$ shares then there is a share $\leq \frac{m}{s} \times \frac{1}{u}$

Example: 5 muffins, 3 students. All student gets $\frac{5}{3}$.
If some student gets $\geq 4$ shares:

Then one of these pieces is $\leq \frac{5}{3} \times \frac{1}{4}$
Pieces Principle

If there are $P$ pieces then:

Some student gets $\geq \left\lceil \frac{P}{s} \right\rceil$

Some student gets $\leq \left\lfloor \frac{P}{s} \right\rfloor$

Example: 5 muffins, 3 people. If there are 10 pieces:

Some student gets $\geq \left\lceil \frac{10}{3} \right\rceil = 4$

Some student gets $\leq \left\lfloor \frac{10}{3} \right\rfloor = 3$
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. *(Negation: All muffins are cut into 2 pieces.)*

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: Someone gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed!

We have many results like this:

\[
f(47, 9) = \frac{111}{234}
\]
\[
f(52, 11) = \frac{83}{176}
\]
\[
f(35, 13) = \frac{64}{143}
\]
General Problem

How can you divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

Let $m, s \in \mathbb{N}$.

An $(m, s)$-procedure is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is optimal if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure.

We have shown $f(5, 3) = \frac{5}{12}$. 
Terminology Issue

Let $m, s \in \mathbb{N}$.  
$m$ is the number of muffins.  
$s$ is the number of students.  

1. $f(m, s) \geq \alpha$ means that there is a procedure with smallest piece $\alpha$. We call this *A Procedure*.

2. $f(m, s) \leq \alpha$ means that there is NO procedure with smallest piece $> \alpha$. We call this *An Optimality Result* or *An Opt Result*.

**DO NOT** use terms *upper bound* and *lower bounds*:

1. Procedures are lower bounds, *opposite* of usual terminology.
2. Opt results are upper bounds, *opposite* of usual terminology.
Floor-Ceiling Theorem

\[ f(m, s) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{m}{s \lfloor 2m/s \rfloor}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}. \]

Proof:

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. Some piece \( \leq \frac{1}{3} \).

**Case 2:** Every muffin is cut into 2 pieces, so 2 \( m \) pieces.

**Someone** gets \( \geq \lfloor \frac{2m}{s} \rfloor \) pieces. Some piece is \( \leq \frac{(m/s)}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor} \).

**Someone** gets \( \leq \lceil \frac{2m}{s} \rceil \) pieces. Some piece is \( \geq \frac{(m/s)}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil} \).

The other piece from that muffin is of size \( \leq 1 - \frac{m}{s \lfloor 2m/s \rfloor} \).
CLEVERNESS, COMP PROGS for the procedure. Floor-Ceiling Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \ k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Is FIVE student case a Mod 5 pattern?

VOTE YES or NO

NO! (excited because YES would be boring)
FIVE Students, $m = 1, 2, 3, 4, 7, 11, 10k$

\[ f(1, 5) = \frac{1}{5} \text{ (easy)} \]
\[ f(2, 5) = \frac{1}{5} \text{ (easy)} \]
\[ f(3, 5) = \frac{1}{4} \text{ (Will discuss briefly later)} \]
\[ f(4, 5) = \frac{3}{10} \text{ (Will not discuss later)} \]
\[ f(7, 5) = \frac{1}{3} \text{ (Use Floor-Ceiling Thm)} \]
\[ f(11, 5) = \text{ (Will come back to this later)} \]
\[ f(10k, 5) = 1 \text{ (Trivial)} \]
FIVE Students

Results on the next few slides:

**CLEVERNESS, COMP PROGS** for the procedure.

**Floor-Ceiling Theorem** for optimality.
FIVE Students \( m = 10k + 1, 10k + 2, 10k + 3 \)

If \( k \) not specified then \( k \geq 0 \).

\( m = 10k + 1: \)

\[ f(30k + 1, 5) = \frac{30k + 1}{60k + 5} \]

\[ f(30k + 11, 5) = \frac{30k + 11}{60k + 25} \quad (k \geq 1) \]

\[ f(30k + 21, 5) = \frac{10k + 7}{20k + 15} \]

\( f(10k + 2, 5) = \frac{10k - 2}{20k} \quad (k \geq 1) \)

\[ f(10k + 3, 5) = \frac{10k + 3}{20k + 10} \quad (k \geq 1) \]
FIVE Students \( m = 10k + 4, 10k + 5, 10k + 6 \)

\[
m = 10k + 4
\]

\[
f(30k + 4, 5) = \frac{30k + 1}{60k + 5}
\]

\[
f(30k + 14, 5) = \frac{30k + 11}{60k + 25}
\]

\[
f(30k + 24, 5) = \frac{10k + 7}{20k + 15}
\]

\[
f(10k + 5, 5) = 1
\]

\[
m = 10k + 6:
\]

\[
f(30k + 6, 5) = \frac{10k + 2}{20k + 5}
\]

\[
f(30k + 16, 5) = \frac{30k + 16}{60k + 35}
\]

\[
f(30k + 26, 5) = \frac{30k + 26}{60k + 55}
\]
FIVE Students \( m = 10k + 7, 10k + 8, 10k + 9 \)

\[
f(10k + 7, 5) = \frac{10k+3}{20k+10}
\]

\[
f(10k + 8, 5) = \frac{5k+4}{10k+10}
\]

\( m = 10k + 9 \)

\[
f(30k + 9, 5) = \frac{10k+2}{20k+5}
\]

\[
f(30k + 19, 5) = \frac{30k+16}{60k+35}
\]

\[
f(30k + 29, 5) = \frac{30k+26}{60k+55}
\]
What About FIVE students, ELEVEN muffins?

Procedure:

Divide the Muffins in to Pieces:

1. Divide 6 muffins into \(\frac{13}{30}, \frac{17}{30}\).
2. Divide 4 muffins into \(\frac{9}{20}, \frac{11}{20}\).
3. Divide 1 muffin into \(\frac{1}{2}, \frac{1}{2}\).

Distribute the Shares to Students:

1. Give 2 students \(\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{1}{2}\).
2. Give 2 students \(\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{9}{20}, \frac{9}{20}\).
3. Give 1 student \(\frac{11}{20}, \frac{11}{20}, \frac{11}{20}, \frac{11}{20}\).

So

\[ f(11, 5) \geq \frac{13}{30} \]
What About FIVE students, ELEVEN muffins? Opt

Recall: **Floor-Ceiling Theorem:**

\[
f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \times 5}, 1 - \frac{11}{5 \times 4} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{25}, \frac{9}{20} \right\} \right\}.
\]

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \frac{11}{25} \right\} = \frac{11}{25}.
\]
Where Are We On FIVE students, ELEVEN muffins?

- By **Procedure** $\frac{13}{30} \leq f(11, 5)$.
- By **Floor-Ceiling** $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666 \ldots$$
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \leq f(11, 5)$.
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

VOTE:

1. **KNOWN**: $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN**: $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN**: $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students for Presidential Election 3 wrote in Harambe.)
Where Are We On FIVE students, ELEVEN muffins?

- By Procedure $\frac{13}{30} \leq f(11, 5)$.
- By Floor-Ceiling $f(11, 5) \leq \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots$$

VOTE:

1. **KNOWN:** $f(11, 5) = \frac{13}{30}$: New opt technique.
2. **KNOWN:** $f(11, 5) = \frac{11}{25}$: New procedure.
3. **KNOWN:** $\frac{13}{30} < f(11, 5) < \frac{11}{25}$: New opt and new proc.
4. **UNKNOWN TO SCIENCE!**
5. **HARAMBE THE GORILLA!**
   (In Poll of Discrete Math Students for Presidential Election 3 wrote in Harambe.)

**KNOWN:** $f(11, 5) = \frac{13}{30}$

**HAPPY:** New opt tech more interesting than new proc.
\( f(11, 5) = \frac{13}{30}, \) Easy Case Based on Muffins

\( N \) is smallest piece.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30}. \)

(*Negation:* All muffins cut into 2 pieces.)
\[ f(11, 5) = \frac{13}{30}, \text{ Easy Case Based on Students} \]

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the shares is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

**(Negation of Cases 2 and 3:** Every student gets 4 or 5 shares.)
\( f(11, 5) = \frac{13}{30}, \) Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 shares
- \( s_5 \) is number of students who get 5 shares

\[
4s_4 + 5s_5 = 22 \\
\]
\[
s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 shares.
\( s_5 = 2 \): There are 2 students who have 5 shares.
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

\[ \diamond \text{ and } \circ \text{ are shares.} \]

\[ \diamond \diamond \diamond \diamond \diamond \quad (\text{Sums to } 11/5) \]
\[ \diamond \diamond \diamond \diamond \diamond \quad (\text{Sums to } 11/5) \]

\[ \circ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]
\[ \circ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]
\[ \circ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]

**Case 3.1:** One of (say)

\[ \circ \circ \circ \circ \circ \quad (\text{Sums to } 11/5) \]

is \( \leq \frac{1}{2} \). Then there is a share

\[ \geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}. \]

The other piece from the muffin is

\[ \leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}. \]
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 3.2:** All

- ○ ○ ○ ○ ○ (Sums to 11/5)
- ○ ○ ○ ○ ○ (Sums to 11/5)
- ○ ○ ○ ○ ○ (Sums to 11/5)

are $> \frac{1}{2}$.
There are $\geq 12$ shares $> \frac{1}{2}$. Can’t occur.
The Techniques Generalizes!

**Good News!**
The technique used to get \( f(11, 5) \leq \frac{13}{30} \) lead to a theorem that apply to other cases! We call it **The Interval Theorem**

**Bad News!**
Interval Theorem is hard to state, so you don’t get to see it.

**Good News!**
Interval Theorem is hard to state, so you don’t have to see it.
How Research Works

**History:**

1. Obtain particular results.
2. Prove a general theorem based on those results.
3. Run into a case we cannot solve (e.g., (11,5) and (35,13)).
4. Lather, Rinse, Repeat.
1. $f(m, s) = \frac{m}{s} f(s, m)$

2. $f(m, s)$ is computable and rational. Uses interesting Applied Math.

3. There is a nice formula for $f(s + 1, s)$. 
What Else Have We Accomplished?-Particular Thms

1. A computer program that helps us get procedures.
2. For $1 \leq s \leq 15$, for all $m$, know $f(m, s)$.
3. Convinced 4 High School students that the most important field of Mathematics is **Muffinry**.
Open Questions

1. For all $s$ there is a pattern for $f(m, s)$ that depends on $m \mod T$ where $s$ divides $T$.

2. $f(m, s) = \frac{a}{b}$ (lowest terms) where $s$ divides $b$.

3. For all $m \geq s$, $f(m, s)$ is always determined by either
   - Floor Ceiling Theorem
   - Interval Theorem
   - $f(s + 1, s)$ Theorem.