Theorem 0.1 \( f(23, 13) = \frac{5}{13} \).

Proof: The following procedure shows \( f(23, 13) \geq \frac{5}{13} \). All numbers have denominator 78 which we omit.

1. Divide 5 muffins (42, 36).
2. Divide 1 muffin (30, 48).
3. Divide 15 muffins (32, 46).
4. Divide 2 muffins (33, 45).
5. Give 5 students [32, 32, 32, 43].
6. Give 1 student [36, 36, 33, 33].
7. Give 5 students [46, 46, 46].
8. Give 1 student [48, 45, 45].

Assume there is a (23, 13)-procedure where the smallest piece is of size > \( \frac{1}{3} \). We can assume every muffin is cut into two pieces and every student gets at least two shares.

Case 1: Some student gets \( \geq 5 \) shares. \( sh_S(5) \leq \frac{23}{13 \times 5} = \frac{23}{65} \leq \frac{5}{13} \).

Case 2: Some student gets \( \leq 2 \) shares. \( B(sh_L(6)) \leq 1 - \frac{23}{13 \times 2} = \frac{3}{26} \leq \frac{5}{13} \).

Case 3: Every muffin is cut in 2 pieces and every student gets either 3 or 4 shares. Note that the total number of shares is 46. Let \( s_3 \) (\( s_4 \)) be the number of 3-students (4-students).

From

\[
3s_3 + 4s_4 = 46
\]
\[
s_3 + s_4 = 13
\]
we derive $s_3 = 6$ and $s_4 = 7$.

**Case 3.1:** There is a 3-share $x \leq \frac{7}{13} \Rightarrow M_L(x) \geq \frac{(23/13) - (7/13)}{2} = \frac{8}{13}$ so $B(M_L(x)) \leq 1 - \frac{8}{13} = \frac{5}{13}$.

**Case 3.2:** There is a 3-share $x \geq \frac{8}{13} \Rightarrow B(x) \leq 1 - \frac{8}{13} = \frac{5}{13}$

**Case 3.3:** All 3-shares are in $(\frac{7}{13}, \frac{8}{13})$ and all 4-shares are in $(\frac{5}{13}, \frac{8}{13})$ (this is not much of a restriction).

We call the 4-shares in $(\frac{5}{13}, \frac{6}{13})$ SM-4 shares (SM for SMall). We chose this interval since all of the buddies of the 3 shares will be SM-4 shares. We call the 4-shares in $[\frac{6}{13}, \frac{7}{13}]$ LA-4 shares (LA for LArge). We call the 4-shares in $(\frac{7}{13}, \frac{8}{13})$ VLA-4 shares (VLA for Very LArge).

The following picture summarizes what we know.

\[
\begin{array}{cccc}
\text{SM-4 shares} & \text{LA-4 shares} & \text{3-shares} & \text{VLA-4-shares} \\
\frac{5}{13} & \frac{6}{13} & \frac{7}{13} & \frac{8}{13}
\end{array}
\]

If the number of SM-4, LA-4, VLA-4 shares is $x,y,z$ then

\[
x + y + z = 28 \\
z + 18 = x \\
z \leq 7
\]

Now to solve

\[
z + 18 + y + z = 28 \\
y + 2z = 10 \\
z = 0, y = 10, x = 18 \\
z = 1, y = 8, x = 19 \\
z = 2, y = 6, x = 20 \\
z = 3, y = 4, x = 21 \\
z = 4, y = 2, x = 22 \\
z = 5, y = 0, x = 23
\]
Claim 1:

1. The number of SM-4 shares equals the sum of the number of 3-shares Plus he number of VLA-4 shares.

2. • Every 4-student has \( \leq 1 \) VLA-4 share
• Every 4-student has \( \geq 2 \) SM-4 share.
• No 4-students has both 1 LA-4 share and 1 VLA-4 share.

3. The only possibilities for how many SM-4 shares, LA-4 shares, VLA-4 shares a 4-share can have are on the following table.

<table>
<thead>
<tr>
<th>SM-4</th>
<th>LA-4</th>
<th>VLA-4</th>
<th>range</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>( \frac{20}{13} &lt; \frac{24}{13} )</td>
<td>YES</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>( \frac{21}{13} &lt; \frac{25}{13} )</td>
<td>YES</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>( \frac{22}{13} &lt; \frac{26}{13} )</td>
<td>YES</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>( \frac{22}{13} &lt; \frac{26}{13} )</td>
<td>YES</td>
</tr>
</tbody>
</table>

4. One of the following two happens:

• Five 4-students have 2 SM-4 shares, zero 4-students have 3 SM-4 shares, two 4-students have 4 SM-4 shares. NOTE- in this case there are 10 LA-4 shares.

• Four 4-students have 2 SM-4 shares, two 4-students have 3 SM-4 shares. one 4-student has 4 SM-4 shares. NOTE- in this case there are

5.

Proof of Claim:
1) If \( x \) is a SM-4-share then \( x < \frac{6}{13} \), so \( B(x) > \frac{7}{11} \), hence \( B(x) \) is a 3-share. If \( y \) is an 3-share then \( x > \frac{7}{13} \), so \( B(x) < \frac{6}{13} \), hence \( B(x) \) is an SM-4 share. Therefore \( B \) is a bijection of SM-4 shares to 3-shares.

2) If a 4-student has \( \geq 2 \) VLA-4 shares then he would have \( > 2 \times 513 + 2 \times 713 = 2413 \).

If a 4-student has 1 SM-4 shares then he has \( > 1 \times \frac{5}{13} + 3 \times \frac{6}{13} = \frac{23}{13} \).

If a 4-students has both 1 LA-4 share and 1 VLA-4 share then he has \( > 2 \times \frac{5}{13} + \frac{6}{13} + \frac{7}{13} = \frac{23}{13} \).

3) Let \( sm_2 \ (sm_3, \ sm_4) \) be the number of 4-students who have 2 (3,4) SM-4 shares. From

\[
2sm_2 + 3sm_3 + 4sm_4 = 18 \\
sm_2 + sm_3 + sm_4 = 7
\]

\((0 \leq sm_2, sm_3, sm_4 \leq 7)\)

we derive

\[
sm_3 + 2sm_4 = 4.
\]

This equation only has two solutions \((0,2)\) and \((2,1)\). Hence the only solutions are \((5,0,2)\) and \((4,2,1)\).

End of Proof of Claim