The Distinct Volumes Problem

David Conlon- Cambridge (Prof)

Jacob Fox-MIT (Prof)

William Gasarch-U of MD (Prof)

David Harris- U of MD (Grad Student)

Douglas Ulrich- U of MD (Ugrad Student)

Sam Zbarsky- Mont. Blair. (High School Student- now CMU)
1. **Infinite Ramsey Theorem:** For any 2-coloring of the EDGES of $K_\omega$ there exists an infinite *monochromatic* $K_\omega$.

2. **Infinite Canonical Ramsey Theorem:** For any $\omega$-coloring of the EDGES of $K_\omega$ there exists an infinite *monochromatic* $K_\omega$ OR an infinite *rainbow* $K_\omega$ OR OTHER STUFF

3. **Want an “application”**. Give an infinite set of points in the plane, color pairs by the distance between.

**Result:** For any infinite set of points in the plane there is an infinite subset where all distances are distinct. (Already known by Erdös via diff proof.)

**Next Step:** Finite version: For every set of $n$ points in the plane there is a subset of size $\Omega(\log n)$ where all distances are distinct. (Much better is known.)
1. Dumped Ramsey approach! Added co-authors! Got new results!

2. What about **Area**? If there are $n$ points in $\mathbb{R}^2$ want large subset so that all areas are distinct.

3. More general question: $n$ points in $\mathbb{R}^d$ and looking for all $a$-volumes to be different. (This question seems to be new.)
The following is an **EXAMPLE** of the kind of theorems we will be talking about. 

*If there are $n$ points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/3})$ with all distances between points DIFF.*
EXAMPLES with AREAS

If there are \( n \) points in \( \mathbb{R}^2 \) then there is a subset of size \( \Omega(n^{1/5}) \) with all triangle areas DIFF.
EXAMPLES with AREAS

If there are $n$ points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas DIFF.

FALSE: Take $n$ points on a LINE. All triangle areas are 0.
If there are $n$ points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas DIFF.

**FALSE:** Take $n$ points on a LINE. All triangle areas are 0.

Two ways to modify:

1. *If there are $n$ points in $\mathbb{R}^2$, no three collinear, then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas DIFF.*

2. *If there are $n$ points in $\mathbb{R}^2$, then there is a subset of size $\Omega(n^{1/5})$ with all nonzero triangle areas DIFF.*

We state theorems in **no three collinear** form.
Maximal Rainbow Sets

**Definition:** A (2)-Rainbow Set is a set of points in $\mathbb{R}^d$ where all of the distances are distinct.

**Definition:** A 3-Rainbow Set is a set of points in $\mathbb{R}^d$ where all nonzero areas of triangles are distinct.

**Definition:** An $a$-Rainbow Set is a set of points in $\mathbb{R}^d$ where all nonzero $a$-volumes are distinct. An $a$-volume is the volume enclosed by $a$ points.

**Definition:** Let $X \subseteq \mathbb{R}^d$. A Maximal Rainbow Set is a rainbow set $Y \subseteq X$ such that if any more points of $X$ are added then it STOPS being a rainbow set.

**Definition:** Let $X \subseteq \mathbb{R}^d$. An $a$-Maximal Rainbow Set is a $a$-rainbow set $Y \subseteq X$ such that if any more points of $X$ are added then it STOPS being an $a$-rainbow set.
Lemma If there is a MAP from $X$ to $Y$ that is $\leq$ c-to-1 then $|Y| \geq |X|/c$.
We will call this LEMMA.
The $d = 1$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^1$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/3})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow M^2 \cup M^3$

What is $f^{-1}(\{x_1, x_2\})$?
The $d = 1$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^1$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/3})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

1. $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
2. $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow M^2 \cup M^3$

What is $f^{-1}({x_1, x_2})$? It’s $\leq 1$ POINT.
The $d = 1$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^1$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/3})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \to M^2 \cup M^3$

What is $f^{-1}(\{x_1, x_2\})$? It's $\leq 1$ POINT.

What is $f^{-1}(\{x_1, x_2, x_3\})$?
The $d = 1$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^1$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/3})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET.**

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \to M^2 \cup M^3$

What is $f^{-1}({x_1, x_2})$? It’s $\leq 1$ POINT.

What is $f^{-1}({x_1, x_2, x_3})$? It’s $\leq 2$ POINTS.
The $d = 1$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^1$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/3})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

$\begin{align*}
\blacktriangleright & \ (\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|].
\blacktriangleright & \ (\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|].
\end{align*}$

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \to M^2 \cup M^3$

What is $f^{-1}(\{x_1, x_2\})$? It's $\leq 1$ POINT.

What is $f^{-1}(\{x_1, x_2, x_3\})$? It's $\leq 2$ POINTS.

$f : X - M \to M^2 \cup M^3$ is $\leq 2$-to-1.
The $d = 1$ Case - Cont

$$f : X - M \rightarrow M^2 \cup M^3 \text{ is } \leq 2\text{-to-1.}$$

**Case 1:** $|M| \geq n^{1/3}$ DONE!

**Case 2:** $|M| \leq n^{1/3}$. So $|X - M| = \Theta(|X|)$. By LEMMA

$$|M^3 + M^2| \geq 0.5|X - M| = \Omega(|X|) = \Omega(n)$$

$$M \geq \Omega(n^{1/3})$$
On Circle

**Theorem:** For all $X \subseteq S^1$ (the circle) of size $n$ there exists a rainbow subset of size $\Omega(n^{1/3})$.

**Proof:** Use **MAXIMAL RAINBOW SET**. Similar Proof.
Better is known: In 1975 Komlos, Sulyok, Szemeredi showed:

**Theorem:** For all $X \subseteq \mathbb{S}^1$ or $\mathbb{R}^1$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/2})$.

This is optimal in $\mathbb{S}^1$ and $\mathbb{R}^1$

**Theorem:** If $X = \{1, \ldots, n\}$ then the largest rainbow subset is of size $\leq (1 + o(1))n^{1/2}$.
The $d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow M^2 \cup M^3$

What is $f^{-1}({x_1, x_2})$?
The $d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow M^2 \cup M^3$

What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.
The $d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow M^2 \cup M^3$

What is $f^{-1}(\{x_1, x_2\})$? Lies on **LINE**.

What is $f^{-1}(\{x_1, x_2, x_3\})$?
The $d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow M^2 \cup M^3$

What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.

What is $f^{-1}(\{x_1, x_2, x_3\})$? Lies on CIRCLE.
The $d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to **reason** $x \not\in M$.

$f : X - M \rightarrow M^2 \cup M^3$

What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.

What is $f^{-1}(\{x_1, x_2, x_3\})$? Lies on CIRCLE.

All INVERSE IMG’s lie on LINES or CIRCLES.
The $d = 2$ Case - Cont

\[ f : X - M \rightarrow M^2 \cup M^3 \]

All INVERSE IMG’s lie on LINES or CIRCLES. $\delta$ TBD.
Cases 1 and 2 induct into line and circle case.

**Case 1:** \((\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^\delta].\)

$\geq n^\delta$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3}).$

**Case 2:** \((\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, x_2, x_3\})| \geq n^\delta].\)

$\geq n^\delta$ points on a circle, so rainbow set size $\geq \Omega(n^{\delta/3}).$

**Case 3:** \(|M| \geq n^{1/6} \text{ DONE!}\)

**Case 4:** Map is $\leq n^\delta$-to-1 AND \(|X - M| = \Theta(|X|).\) By LEMMA

\[
|M^2 \cup M^3| \geq n/n^\delta = n^{1-\delta} \quad |M| \geq \Omega(n^{(1-\delta)/3})
\]

Set $\delta/3 = (1 - \delta)/3$. $\delta = 1/2$. Get $\Omega(n^{1/6}).$
**On Sphere**

**Theorem:** For all $X \subseteq S^2$ (surface of sphere) of size $n$ there exists a rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Use **MAXIMAL RAINBOW SET**. Similar Proof.

**Note:** Better is known: Charalambides showed $\Omega(n^{1/3})$. 
General $d$ Case

**Theorem:**
For all $X \subseteq \mathbb{R}^d$ of size $n$ $\exists$ rainbow subset of size $\Omega(n^{1/3d})$.
For all $X \subseteq \mathbb{S}^d$ of size $n$ $\exists$ rainbow subset of size $\Omega(n^{1/3d})$.

**Proof:** Use **MAXIMAL RAINBOW SET** and induction. Need result on $\mathbb{S}^d$ and $\mathbb{R}^d$ to get result for $\mathbb{S}^{d+1}$ and $\mathbb{R}^{d+1}$.

**Note:** Better is known. In 1995 Thiele showed $\Omega(n^{1/(3d-2)})$. But WE improved that!
**Theorem:** For all $d \geq 2$, for all $X \subseteq \mathbb{R}^d$ of size $n$ there exists a rainbow subset of size $\Omega(n^{1/(3d-3)}(\log n)^{1/3 - 2/(3d-3)})$.

**Proof:** Use **VARIANT ON MAX RAINBOW SET**

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n^{1/3d}$</th>
<th>$n^{1/(3d-3)}(\log n)^{1/3 - 2/(3d-3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$n^{1/3}$</td>
<td>--</td>
</tr>
<tr>
<td>2</td>
<td>$n^{1/6}$</td>
<td>$n^{1/3}(\log n)^{-1/3}$</td>
</tr>
<tr>
<td>3</td>
<td>$n^{1/9}$</td>
<td>$n^{1/6}(\log n)^0$</td>
</tr>
<tr>
<td>4</td>
<td>$n^{1/12}$</td>
<td>$n^{1/9}(\log n)^{1/12}$</td>
</tr>
<tr>
<td>5</td>
<td>$n^{1/15}$</td>
<td>$n^{1/12}(\log n)^{1/6}$</td>
</tr>
<tr>
<td>6</td>
<td>$n^{1/18}$</td>
<td>$n^{1/15}(\log n)^{1/5}$</td>
</tr>
</tbody>
</table>

Can we do better? Best we can hope for is roughly $n^{1/d}$. 
Theorem: For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ 3-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let $M$ be a **MAXIMAL RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to reason $x \not\in M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$. Recall that

What is $f^{-1}(\{x_1, x_2, x_3\})$? SEE NEXT SLIDE FOR GEOM LEMMA.
Lemma: Let $L_1$ and $L_2$ be lines in $\mathbb{R}^2$. 

$$\{ p : \text{AREA}(L_1, p) = \text{AREA}(L_2, p) \}$$

is a line.

Sketch: $\text{AREA}(L_1, p) = \text{AREA}(L_2, p)$ iff 

$$|L_1| \times |L_1 - p| = |L_2| \times |L_2 - p| \text{ iff } \frac{|L_1 - p|}{|L_2 - p|} = \frac{|L_1|}{|L_2|}. \text{ This is a line.}$$
(Reboot) Area-\(d = 2\) Case

**Theorem:** For all \(X \subseteq \mathbb{R}^2\) of size \(n\), no three colinear, \(\exists\) 3-rainbow set of size \(\Omega(n^{1/5})\).

**Proof:** Let \(M\) be a **MAXIMAL RAINBOW SET**.

Let \(x \in X - M\). WHY IS \(x\) NOT IN \(M\)!? Either

- \((\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]\).
- \((\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]\).
- \((\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]\).

\(f\) maps an element of \(X - M\) to **reason** \(x \notin M\).

\(f : X - M \rightarrow M^3 \cup M^4 \cup M^5\). Recall that
(Reboot) Area-$d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ 3-rainbow set of size $\Omega(n^{1/5})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$. Recall that

What is $f^{-1}(\{x_1, x_2, x_3, x_4\})$?
(Reboot) Area-$d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ 3-rainbow set of size $\Omega(n^{1/5})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.
Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either
- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$. Recall that
What is $f^{-1}(\{x_1, x_2, x_3, x_4\})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE.**
(Reboot) Area- \(d = 2\) Case

**Theorem:** For all \(X \subseteq \mathbb{R}^2\) of size \(n\), no three colinear, \(\exists 3\)-rainbow set of size \(\Omega(n^{1/5})\).

**Proof:** Let \(M\) be a **MAXIMAL RAINBOW SET**.

Let \(x \in X - M\). WHY IS \(x\) NOT IN \(M\)!? Either

- \((\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]\).
- \((\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]\).
- \((\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]\).

\(f\) maps an element of \(X - M\) to reason \(x \notin M\).

\(f: X - M \to M^3 \cup M^4 \cup M^5\). Recall that

What is \(f^{-1}(\{x_1, x_2, x_3, x_4\})\)? By Lemma all points on it are on a line- so \(\leq 2\) points. **FINITE.**

What is \(f^{-1}(\{x_1, x_2, x_3, x_4\})\)?
(Reboot) Area-$d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ 3-rainbow set of size $\Omega(n^{1/5})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.
Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f: X - M \rightarrow M^3 \cup M^4 \cup M^5$. Recall that

What is $f^{-1}(\{x_1, x_2, x_3, x_4\})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE**.

What is $f^{-1}(\{x_1, x_2, x_3, x_4\})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE**.
(Reboot) Area-$d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ 3-rainbow set of size $\Omega(n^{1/5})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$. Recall that

What is $f^{-1}({x_1, x_2, x_3, x_4})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE**.

What is $f^{-1}({x_1, x_2, x_3, x_4})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE**.

What is $f^{-1}({x_1, x_2, x_3, x_4, x_5})$?
(Reboot) Area-$d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ 3-rainbow set of size $\Omega(n^{1/5})$.

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)].$
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)].$
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)].$

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$. Recall that

What is $f^{-1}({x_1, x_2, x_3, x_4})$? By Lemma all points on it are on a line- so $\leq 2$ points. FINITE.

What is $f^{-1}({x_1, x_2, x_3, x_4})$? By Lemma all points on it are on a line- so $\leq 2$ points. FINITE.

What is $f^{-1}({x_1, x_2, x_3, x_4, x_5})$? By Lemma all points on it are on a line- so $\leq 2$ points. FINITE.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$ FINITE-to-1.
Area $d = 2$ Case - Cont

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$ is FINITE-to-1.

**Case 1:** $|M| \geq n^{1/5}$ DONE!

**Case 2:** $|M| \leq n^{1/5}$. Then $|X - M| = \Theta(|X|)$. Since MAP is finite-to-1, by LEMMA

\[
\begin{align*}
|M^3 \cup M^2 \cup M^5| & \geq \Omega(|X - M|) = \Omega(|X|) = \Omega(n) \\
|M| & \geq \Omega(n^{1/5})
\end{align*}
\]
KEY to These Proofs

1. Used **MAXIMAL RAINBOW SET** $M$.
2. Used Map $f$ from $x \in X - M$ to the reason $x$ is NOT in $M$.
3. Looked at **INVERSE IMAGES** of that map.
4. Either:
   - All INVERSE IMG’s are small, so use LEMMA.
   - OR
   - Some INVERSE IMG’s are large subsets of $\mathbb{R}^d$ or $S^d$, so induct.
**Area-$d = 3$ Case**

**Theorem:** For all $X \subseteq \mathbb{R}^3$ of size $n$, there exists 3-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.
Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\operatorname{AREA}(x, x_1, x_2) = \operatorname{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[A(x, x_1, x_2) = \operatorname{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\operatorname{AREA}(x, x_1, x_2) = \operatorname{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$.

What is $f^{-1}(\{x_1, x_2, x_3\})$?
**Area-$$d = 3$$ Case**

**Theorem:** For all $$X \subseteq \mathbb{R}^3$$ of size $$n$$, there exists 3-rainbow set of size $$\Omega(n^\delta)$$. ($$\delta$$ TBD)

**Proof:** Let $$M$$ be a **MAXIMAL RAINBOW SET**.

Let $$x \in X - M$$. WHY IS $$x$$ NOT IN $$M$$!? Either

- $$(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$$.
- $$(\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$$.
- $$(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$$.

$$f$$ maps an element of $$X - M$$ to **reason** $$x \notin M$$.

$$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$$.

What is $$f^{-1}({x_1, x_2, x_3})$$? THIS IS HARD!
Area-\(d = 3\) Case

**Theorem:** For all \(X \subseteq \mathbb{R}^3\) of size \(n\), there exists 3-rainbow set of size \(\Omega(n^\delta)\). (\(\delta\) TBD)

**Proof:** Let \(M\) be a **MAXIMAL RAINBOW SET**.

Let \(x \in X - M\). Why is \(x\) NOT IN \(M\)? Either

- \((\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]\).
- \((\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]\).
- \((\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]\).

\(f\) maps an element of \(X - M\) to reason \(x \notin M\).

\(f : X - M \rightarrow M^3 \cup M^4 \cup M^5\).

What is \(f^{-1}(\{x_1, x_2, x_3\})\)? THIS IS HARD!

What is \(f^{-1}(\{x_1, x_2, x_3, x_4\})\)?
Area-$d = 3$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^3$ of size $n$, there exists a 3-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[A(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \to M^3 \cup M^4 \cup M^5$.

What is $f^{-1}(\{x_1, x_2, x_3\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2, x_3, x_4\})$? THIS IS HARD!
Area-$d = 3$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^3$ of size $n$, there exists 3-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$.

What is $f^{-1}(\{x_1, x_2, x_3\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2, x_3, x_4\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2, x_3, x_4, x_5\})$?
Area-$d = 3$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^3$ of size $n$, there exists 3-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)

**Proof:** Let $M$ be a **MAXIMAL RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow M^3 \cup M^4 \cup M^5$.

What is $f^{-1}(\{x_1, x_2, x_3\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2, x_3, x_4\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2, x_3, x_4, x_5\})$? THIS IS HARD!

What to do?
WHAT CHANGED?

Why is this proof harder?

**KEY** statement about prior proof:

1. If INVERSE IMG’s are all finite so $M$ is large.
2. If INVERSE IMG’s are subsets of $\mathbb{R}^d$ or $\mathbb{S}^d$ then induct.

**KEY:** We cared about $X \subseteq \mathbb{R}^d$ but had to work with $\mathbb{S}^d$ as well. NOW we will have to work with more complicated objects.
What Do Inverse Images Look Like?

\[ \{ x : \text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4) \} = \]

\[ \{ x : |\text{DET}(x, x_1, x_2)| = |\text{DET}(x, x_3, x_4)| \}. \]

**Definition:** (Informally) An **Algebraic Variety in** \( \mathbb{R}^d \) **is a set of points in** \( \mathbb{R}^d \) **that satisfy a polynomial equation in** \( d \) **variables.**
General Theorem

**Theorem** Let $2 \leq a \leq d + 1$. Let $r \in \mathbb{N}$. For all varieties $V$ of dim $d$ and degree $r$ for all sets of $n$ points on $V$ there exists an $a$-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

**Corollary** Let $2 \leq a \leq d + 1$. For all $X \subseteq \mathbb{R}^d$ of size $n$ there exists an $a$-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

**Corollary** For all $X \subseteq \mathbb{R}^d$ of size $n$ there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

**Corollary** For all $X \subseteq \mathbb{R}^d$ of size $n$ there is a 3-rainbow set (dist. areas) of size $\Omega(n^{1/5d})$.

**Corollary** For all $X \subseteq \mathbb{R}^d$ of size $n$ there is a 4-rainbow set (dist. volumes) of size $\Omega(n^{1/7d})$.

**Comments on the Proof**

1. Proof uses Algebraic Geometry in Proj Space over $\mathbb{C}$.
2. Proof uses Maximal subsets in same way as easier proofs.
3. Proof is by induction on $d$. 
Open Questions

1. Better Particular Results: e.g., want
   for all $X \subseteq \mathbb{R}^2$ of size $n$, there exists a rainbow set of size
   $\Omega(n^{1/2})$.

2. General Better Results: e.g., want
   Let $1 \leq a \leq d + 1$. For all $X \subseteq \mathbb{R}^d$ of size $n$ there exists a
   rainbow set of size $\Omega(n^{1/ad})$.

3. Get easier proofs of general theorem.

4. Find any nontrivial limits on what we can do. (Trivial: $n^{1/d}$).

5. Algorithmic aspects.