Constructions in Computable Ramsey Theory
(An Exposition)

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Notation

Notation:

1. \( M_1, M_2, \ldots \) is a standard list of Turing Machines.
2. Note that from \( e \) we can extract the code for \( M_e \).
3. \( M_{e,s}(x) \) means that we run \( M_e \) for \( s \) steps.
4. \( W_e \) is the domain of \( M_e \), that is,
   \[
   W_e = \{ x \mid (\exists s)[M_{e,s}(x) \downarrow] \}.
   \]
   Note that \( W_1, W_2, \ldots \) is a list of ALL c.e. sets.
5. \( W_{e,s} = \{ x \mid M_{e,s}(x) \downarrow \} \).
There exists computable $COL : \binom{\mathbb{N}}{2} \rightarrow [2]$ such that there is NO infinite c.e. homog set.
We construct $COL : \binom{\mathbb{N}}{2} \to [2]$ to satisfy:

$$R_e : W_e \text{ infinite} \implies W_e \text{ NOT a homog set}.$$ 

**CONSTRUCTION OF COLORING**

*Stage 0:* $COL$ is not defined on anything.

*Stage s:* We will define $COL(0, s), COL(1, s), \ldots, COL(s - 1, s)$.

For all $0 \leq e \leq s$ do the following, starting with $e = 0$:

If $(\exists x, y \leq s - 1)[x, y \in W_{e,s} \land COL(x, s), COL(y, s) \text{ undefined}]$ then define take LEAST such $x, y$ and do:

1. $COL(x, s) = RED$,
2. $COL(y, s) = BLUE$. (Note that IF $s \in W_e$ then $R_e$ would be satisfied.)

After all this, for all $(x, s)$ not yet colored, $COL(x, s) = RED$.

**END OF CONSTRUCTION**
There is a Comp Coloring with no Inf c.e.-in-HALT Homog Set

**Theorem**

There exists computable \( \text{COL} : \binom{N}{2} \rightarrow [2] \) such that there is no infinite c.e-in-HALT homog set.

This is on HW1.
Every Comp Coloring has inf $\Pi_2$ Homog Set

Theorem

For every computable coloring $COL : \binom{\mathbb{N}}{2} \to [2]$ there is an infinite $\Pi_2$ homog set.
Construction of Inf $\Pi_2$ Homog Set

Given computable $COL : (\mathbb{N}_2) \to [2]$.

**CONSTRUCTION of** $x_1, x_2, \ldots$ **and** $c_1, c_2, \ldots$.

$x_1 = x$ and $c_1 = RED$ (We are guessing. Might change later)

$s \geq 2$, assume $x_1, \ldots, x_{s-1}$ and $c_1, \ldots, c_{s-1}$ are defined.

Ask $HALT \left( (\exists x \geq x_{s-1}) (\forall 1 \leq i \leq s - 1) [COL(x_i, x) = c_i] \right)$?

**YES:** Find least such $x$.

- $x_i = x$
- $c_i = RED$ (Guessing.)
Construction of Inf $\Pi_2$ Homog Set: NO Case

NO: Ask $HALT$:

1. $(\exists x \geq x_{s-1})(\forall 1 \leq i \leq s-2)[COL(x_i, x) = c_i]$?
2. $(\exists x \geq x_{s-1})(\forall 1 \leq i \leq 1)[COL(x_i, x) = c_i]$?

Let $i_0$ be largest such that

$(\exists x \geq x_{s-1})(\forall 1 \leq i \leq i_0)[COL(x_i, x) = c_i]$?

1. Change color of $c_{i+1}$.
2. Wipe out $x_{i+2}, \ldots, x_{s-1}$.
3. Find $x \geq x_{s-1}$ such that $(\forall 1 \leq i \leq i_0)[COL(x_i, x) = c_i]$
4. $x_{i+2} = x$. $c_{i+2} = RED$ (Guessing)

END OF CONSTRUCTION of $x_1, x_2 \ldots$ and $c_1, c_2, \ldots$
Getting the Inf $\Pi_2$ Homog Set

$X = \{ x_1, x_2, \ldots \}$. $R$ is the set of red elts of $X$

$\overline{X} \in \Sigma_2$ (so $X \in \Pi_2$).

$\overline{X} = \{ x \mid (\exists s)[ \text{at stage } s \text{ of the construction } x \text{ was tossed out} ] \}$.

$\overline{R} \in \Sigma_2$ (so $R \in \Pi_2$).

$\overline{R} = \overline{X} \cup \{ x \mid (\exists x)[ \text{at stage } s \text{ of the construction } x \text{ was turned BLUE} ] \}$.

1. If $R$ is infinite then $R$ is inf homog set in $\Pi_2$.
2. If $R$ is finite then $B = X - R$ is inf homog set in $\Pi_2$. 