Computability Theory and Ramsey Theory
An Exposition by William Gasarch

All of the results in this document are due to Jockusch [?].

1 A Computable Coloring with NO Infinite c.e. Homog Sets

All of the results in this

**Notation 1.1**

1. $M_1, M_2, \ldots$ is a standard list of Turing Machines.

2. Note that from $e$ we can extract the code for $M_e$.

3. $M_{e,s}(x)$ means that we run $M_e$ for $s$ steps.

4. $W_e$ is the domain of $M_e$, that is,

$$W_e = \{x \mid (\exists s)[M_{e,s}(x) \downarrow]\}.$$  

Note that $W_1, W_2, \ldots$ is a list of ALL c.e. sets.

5. 

$$W_{e,s} = \{x \mid M_{e,s}(x) \downarrow\}.$$  

**Theorem 1.2** There exists a computable $COL : \binom{\mathbb{N}}{2} \rightarrow [2]$ such that there is NO infinite c.e. homog set.

**Proof:** We construct $COL : \binom{\mathbb{N}}{2} \rightarrow [2]$ to satisfy the following requirements (NOTE- requirements is the most important word in computability theory.)
$R_e : W_e \text{ infinite } \implies W_e \text{ NOT a homog set}.$

**CONSTRUCTION OF COLORING**

*Stage 0:* $COL$ is not defined on anything.

*Stage $s$:* We define $COL(0, s), \ldots, COL(s - 1, s)$. For $e = 0, 1, \ldots, s$:

If this occurs:

\[(\exists x, y \leq s - 1)[x, y \in W_{e,s} \land COL(x, s), COL(y, s) \text{ undefined}]\]

then take the LEAST two $x, y$ for which this is the case and do the following:

- $COL(x, s) = RED$
- $COL(y, s) = BLUE$.

(Note that IF $s \in W_e$ (which we do not know at this time) then $R_e$ would be satisfied.)

After you to through all of the $0 \leq e \leq s$ define all other $COL(x, y)$ where $0 \leq x < y \leq s$ that have not been defined by $COL(x, y) = RED$. This is arbitrary. The important things is that ALL $COL(x, s)$ where $0 \leq x \leq s - 1$ are now defined. This is why $COL$ is computable— at stage $s$ we have defined all $COL(x, y)$ with $0 \leq x < y \leq s$.

**END OF CONSTRUCTION**

We show that each requirement is eventually satisfied.

For pedagogue we first look at $R_1$.

If $W_1$ is finite then $R_1$ is satisfied.

Assume $W_1$ is infinite. We show that $R_1$ is satisfied. Let $x < y$ be the least two elements in $W_1$. Let $s_0$ be the least number such that $x, y \in W_{1,s_0}$ Note that, for ALL $s \geq s_0$ you will have $COL(x, s) = RED$
\[ \text{COL}(y,s) = \text{BLUE} \]

Since \( W_1 \) is infinite there is SOME \( s \geq s_0 \) with \( s \in W_e \). Hence \( x, y, s \in W_1 \) and show that \( W_1 \) is NOT homogenous.

Can we show \( R_2 \) is satisfied the same way? Yes but with a caveat- we won’t use the least two elements of \( W_2 \). We’ll use the least two elements of \( W_2 \) that are bigger than the least two elements of \( W_1 \). We now do this rigorously and more generally.

**Claim: For all \( e \), \( R_e \) is satisfied:**

**Proof:** Fix \( e \). If \( W_e \) is finite then \( R_e \) is satisfied.

Assume \( W_e \) is infinite. We show that \( R_e \) is satisfied. Let \( x_1 < x_2 < \cdots < x_{2e} \) be the first (numerically) \( 2e \) elements of \( W_e \). Let \( s_0 \) be the least number such that

- \( x_1, \ldots, x_{2e} \in W_{e,s_0} \), and
- For all \( x \in \{x_1, \ldots, x_{2e}\} \), for all \( 1 \leq i \leq e - 1 \), if \( x \in W_i \) then \( x \in W_{i,s_0} \).

**KEY:** for all \( s \geq s_0 \), during stage \( s \), the requirements \( R_1, \ldots, R_{e-1} \) may define \( \text{COL}(x,s) \) for some of the \( x \in \{x_1, \ldots, x_{2e}\} \). But they will NOT define \( \text{COL}(x,s) \) for ALL of those \( x \). Why? Because \( R_i \) only defines \( \text{COL}(x,s) \) for at most TWO of those \( x \)'s, and there are \( e - 1 \) such \( i \), so at most \( 2e - 2 \) of those \( x \)'s have \( \text{COL}(x,s) \) defined. Hence there will exist \( x, y \) such that \( R_e \) gets to define \( \text{COL}(x,s) \) and \( \text{COL}(y,s) \). Furthermore, they will always be the SAME \( x, y \) since the \( R_i \) with \( i < e \) have already made up their minds about the \( x \) in \( \{x_1, \ldots, x_{2e}\} \).

**UPSHOT:** There exists \( x, y \in W_e \) such that, for all \( s \geq s_0 \),

\[
\text{COL}(x,s) = \text{RED} \\
\text{COL}(y,s) = \text{BLUE}
\]

Since \( W_e \) is infinite there is SOME \( s \geq s_0 \) with \( s \in W_e \). Hence \( x, y, s \in W_e \) and show that \( W_e \) is NOT homogenous.
2 A Computable Coloring with NO c.e.-in-K Homog Sets

Notation 2.1

1. If \( A \) is a c.e. set, say \( A \) is the domain of \( M \), then \( A_s \) is \( \{ x \leq s \mid M_{e,s}(x) \downarrow \} \). Note that, given \( s \), one can compute \( A_s \).

2. \( M_1^{(0)}, M_2^{(0)}, \ldots \) is a standard list of oracle Turing Machines.

3. Note that from \( e \) we can extract the code for \( M_e^{(0)} \).

4. If \( A \) is a c.e. set then \( M_{e,s}^A(x) \) means that we run \( M_e^{(0)} \) for \( s \) steps and using \( A_s \) for the oracle.

5. If \( A \) is c.e. then \( W_e^A \) is the domain of \( M_e^A \).

\[
W_e^A = \{ x \mid (\exists s)[M_{e,s}^A(x) \downarrow] \}.
\]

Note that \( W_1^K, W_2^K, \ldots \) is a list of ALL c.e-in-K sets.

6.

\[
W_{e,s}^{A_s} = \{ x \mid M_{e,s}^{A_s}(x) \downarrow \}.
\]

Theorem 2.2 There exists \( COL : \binom{N}{2} \rightarrow [2] \) such that there is NO infinite c.e-in-K homog set.

Proof sketch: This will be a HW. But note that its very similar to the proof of Theorem 1.2— if \( W_e^K \) is infinite then eventually \( W_{e,s}^{K^e} \) will settle down on its first \( 2e \) elements.

3 A Computable Coloring with NO \( \Sigma_2 \) Homog Sets

We state equivalences of both c.e. and c.e.-in-K. We leave the proofs to the reader.

Theorem 3.1 Let \( A \) be a set. The following are equivalent:
1. There exists $e$ such that $A = W_e$. ($A$ is c.e.)

2. There exists a decidable $R$ such that

   $$A = \{ x \mid (\exists y)(x, y) \in R \}. $$

   ($A$ is $\Sigma^1_1$.)

3. There exists $e$ such that

   $$A = \{ x \mid (\exists y, s)[M_{e,s}^{K}(y) = x] \}. $$

   (This is the origin of the phrase ‘computably ENUMERABLE’.)

**Theorem 3.2** Let $A$ be a set. The following are equivalent:

1. There exists $e$ such that $A = W_e^K$. ($A$ is c.e.-in-$K$.)

2. There exists a decidable-in-$K$ $R$ such that

   $$A = \{ x \mid (\exists y)(x, y) \in R \}. $$

   ($A$ is $\Sigma^K_1$.)

3. There exists $e$ such that

   $$A = \{ x \mid (\exists y, s)[M_{e,s}^{K}(y) = x] \}. $$

   (This is the origin of the phrase ‘computably ENUMERABLE-in-$K$’.)

We also need to know that $K$ is quite powerful:

**Def 3.3** If $A, B$ are sets then $A \leq_m B$ means that there exists a computable $f$ such that

   $$x \in A \iff f(x) \in B.$$
We leave the proof of the following to the reader.

**Theorem 3.4** If $A$ is c.e. then $A \leq_m K$.

The key use of the above theorem is that we can phrase $\Sigma_1$ questions as queries to $K$.

**Theorem 3.5** $A \in \Sigma_2$ iff $A$ is c.e.-in-$K$.

**Proof:**

1) $A \in \Sigma_2$ implies $A$ is c.e.-in-$K$:

   If $A \in \Sigma_2$ then there exists a TM $R$ that always converges such that

   $$A = \{ x \mid (\exists y)(\forall z)[R(x, y, z) = 1] \}.$$ 

   Let $M^K$ be the TM that does the following:

   1. Input($x, y$).
   2. Ask $K$ $(\forall z)[R(x, y, z) = 1]$. (Can rephrase as $(\exists z)[R(x, y, z) = 0]$.)
   3. If YES answer YES, if NO then answer NO.

   $$A = \{ x \mid (\exists y)[M^K(x, y) = 1] \}.$$ 

   Hence $A$ is c.e.-in-$K$.

2) $A$ c.e.-in-$K$ implies $A \in \Sigma_2$.

   $A$ is c.e.-in-$K$. So

   $$A = W^K_e = \{ x \mid (\exists s)(\forall t)[t \geq s \implies x \in W^K_{e,t}] \}.$$ 

   So $A$ is $\Sigma_2$. 

\[\]
Theorem 3.6  There exists $COL : \binom{\mathbb{N}}{2} \rightarrow [2]$ such that there is NO infinite $\Sigma_2$ homog set.

Proof:  Combine Theorems 2.2 and 3.5. Note that we only need one part of the implication in Theorem 3.5.  

4 Every Computable Coloring has an Infinite $\Pi_2$ Homog set

We obtain this with a modification of the usual proof of Ramsey’s theorem. the key is that we don’t really toss things out- we guess on what the colors are and change our mind.

Theorem 4.1  For every computable coloring $COL : \binom{\mathbb{N}}{2} \rightarrow [2]$ there is an infinite $\Pi_2$ homog set.

Proof:  
We are given computable $COL : \binom{\mathbb{N}}{2} \rightarrow [2]$.  
CONSTRUCTION of $x_1, x_2, \ldots$ and $c_1, c_2, \ldots$  

NOTE: at the end of stage $s$ we might have $x_1, \ldots, x_i$ defined where $i < s$. We will not try to keep track of how big $i$ is. Also, we may have at stage (say) 1000 a sequence of length 50, and then at stage 1001 have a sequence of length only 25. The sequence will grow eventually but do so in fits and starts.

$x_1 = 1$

c_1 = \text{RED}  \text{ We are guessing. We might change our mind later}$

Let $s \geq 2$, and assume that $x_1, \ldots, x_{s-1}$ and $c_1, \ldots, c_{s-1}$ are defined.

1. Ask $K$ Does there exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq s-1$, $COL(x_i, x) = c_i$?

7
2. If YES then (using that \(\text{COL}\) is computable) find the least such \(x\).

\[ x_i = x \]

\[ c_i = \text{RED} \quad \text{We are guessing. We might change our mind later} \]

We have implicitly tossed out all of the numbers between \(x_{i-1}\) and \(x_i\).

3. If NO then we ask \(K\) how far back we can go. More rigorously we ask the following sequence of questions until we get a YES.

- \(\text{Does there exists } x \geq x_{s-1} \text{ such that, for all } 1 \leq i \leq s - 2, \text{COL}(x_i, x) = c_i?\)
- \(\text{Does there exists } x \geq x_{s-1} \text{ such that, for all } 1 \leq i \leq s - 3, \text{COL}(x_i, x) = c_i?\)
- \(\vdots\)
- \(\text{Does there exists } x \geq x_{s-1} \text{ such that, for all } 1 \leq i \leq 2, \text{COL}(x_i, x) = c_i?\)
- \(\text{Does there exists } x \geq x_{s-1} \text{ such that, for all } 1 \leq i \leq 1, \text{COL}(x_i, x) = c_i?\)

(One of these must be a YES since (1) if \(c_1 = \text{RED}\) and there are NO red edges coming out of \(x_1\) then there must be an infinite number of \(\text{BLUE}\) edges, and (2) if \(c_1=\text{BLUE}\) its because there are only a finite number of \(\text{RED}\) edges coming out of \(x_1\) so there are an infinite number of \(\text{BLUE}\) edges. Let \(i_0\) be such that \(\text{There exists } x \geq x_{s-1} \text{ such that, for all } 1 \leq i \leq i_0, \text{COL}(x_i, x) = c_i\)\) Do the following:

(a) Change the color of \(c_{i+1}\). (We will later see that this change must have been from \(\text{RED}\) to \(\text{BLUE}\).
(b) Wipe out \(x_{i+2}, \ldots, x_{s-1}\).
(c) Search for the \(x \geq x_{s-1}\) that the question asked says exist.
(d) \(x_{i+2}\) is now \(x\).
(e) $c_{i+2}$ is now $RED$.

END OF CONSTRUCTION of $x_1, x_2 \ldots$ and $c_1, c_2, \ldots$.

We need to show that there is a $\Pi_2$ homog set.

Let $X$ be the set of $x_i$ that are put on the board and stay on the board.

Let $R$ be the set of $x_i \in X$ whose final color is $RED$.

**Claim 1:** Once a number turns from $RED$ to $BLUE$ it can't go back to $RED$ again.

**Proof:**

If a number is turned $BLUE$ its because there are only a finite number of $RED$ edges coming out of it. Hence there must be an infinite number of $BLUE$ edges coming out of it. Hence it will never change color (though it may be tossed out).

**End of Proof**

**Claim 1:** $X, R \in \Pi_2$.

**Proof:**

We show that $\overline{X} \in \Sigma_2$. In order to NOT be in $X$ you must have, at some point in the construction, been tossed out.

$$\overline{X} = \{ x \mid (\exists x)[ \text{at stage } s \text{ of the construction } x \text{ was tossed out}] \}.$$  

Note that the condition is computable-in-$K$. Hence $\overline{X}$ is c.e.-in-$K$. By Theorem 3.5 $\overline{X} \in \Sigma_2$.

We show that $\overline{R} \in \Sigma_2$. In order to NOT be in $R$ you must have to either NOT be in $X$ or have been turned blue. Note that once you turn at some point in the construction, been tossed out.

$$\overline{R} = \overline{X} \cup \{ x \mid (\exists x)[ \text{at stage } s \text{ of the construction } x \text{ was turned BLUE}] \}.$$  

Recall that $\Sigma_2$ is closed under complementation. So we only need to show that the other unio-nand is in $\Sigma_2$. Note that the condition is computable-in-$K$. Hence $\overline{R}$ is c.e.-in-$K$. By Theorem 3.5
$\overline{R} \in \Sigma_2$. 

**End of Proof**

There are two cases:

1. If $R$ is infinite then $R$ is an infinite homog set that is $\Pi_2$.

2. If $R$ is finite then $B$ is $X$ minus a finite number of elements. Since $X$ is $\Pi_2$, $B$ is $\Pi_2$.

**References**