Computability Theory and Ramsey Theory
An Exposition by William Gasarch

All of the results in this document are due to Jockusch [1].

1 A Computable Coloring with NO Infinite c.e. Homog Sets

All of the results in this

Notation 1.1

1. \( M_1, M_2, \ldots \) is a standard list of Turing Machines.

2. Note that from \( e \) we can extract the code for \( M_e \).

3. \( M_{e,s}(x) \) means that we run \( M_e \) for \( s \) steps.

4. \( W_e \) is the domain of \( M_e \), that is,

\[
W_e = \{ x \mid (\exists s)[M_{e,s}(x) \downarrow] \}.
\]

Note that \( W_1, W_2, \ldots \) is a list of ALL c.e. sets.

5. \( W_{e,s} = \{ x \mid M_{e,s}(x) \downarrow \} \).

Theorem 1.2 There exists a computable \( COL : \binom{\mathbb{N}}{2} \rightarrow [2] \) such that there is NO infinite c.e. homog set.

Proof: We construct \( COL : \binom{\mathbb{N}}{2} \rightarrow [2] \) to satisfy the following requirements (NOTE- requirements is the most important word in computability theory.)
\[ R_e : W_e \text{ infinite } \implies W_e \text{ NOT a homog set.} \]

**CONSTRUCTION OF COLORING**

*Stage 0:* \( \text{COL} \) is not defined on anything.

*Stage \( s \):* We define \( \text{COL}(0, s), \ldots, \text{COL}(s - 1, s) \). For \( e = 0, 1, \ldots, s \):

If this occurs:

\[(\exists x, y \leq s - 1) [x, y \in W_e, s \land \text{COL}(x, s), \text{COL}(y, s) \text{ undefined}]\]

then take the LEAST two \( x, y \) for which this is the case and do the following:

- \( \text{COL}(x, s) = \text{RED} \)
- \( \text{COL}(y, s) = \text{BLUE} \).

(Note that IF \( s \in W_e \) (which we do not know at this time) then \( R_e \) would be satisfied.)

After you to through all of the \( 0 \leq e \leq s \) define all other \( \text{COL}(x, y) \) where \( 0 \leq x < y \leq s \) that have not been defined by \( \text{COL}(x, y) = \text{RED} \). This is arbitrary. The important things is that ALL \( \text{COL}(x, s) \) where \( 0 \leq x \leq s - 1 \) are now defined. This is why \( \text{COL} \) is computable— at stage \( s \) we have defined all \( \text{COL}(x, y) \) with \( 0 \leq x < y \leq s \).

**END OF CONSTRUCTION**

We show that each requirement is eventually satisfied.

For pedagogue we first look at \( R_1 \).

If \( W_1 \) is finite then \( R_1 \) is satisfied.

Assume \( W_1 \) is infinite. We show that \( R_1 \) is satisfied. Let \( x < y \) be the least two elements in \( W_1 \). Let \( s_0 \) be the least number such that \( x, y \in W_{1, s_0} \). Note that, for ALL \( s \geq s_0 \) you will have \( \text{COL}(x, s) = \text{RED} \)
\( \text{COL}(y, s) = \text{BLUE} \)

Since \( W_1 \) is infinite there is SOME \( s \geq s_0 \) with \( s \in W_e \). Hence \( x, y, s \in W_1 \) and show that \( W_1 \) is NOT homogenous.

Can we show \( R_2 \) is satisfied the same way? Yes but with a caveat- we won’t use the least two elements of \( W_2 \). We’ll use the least two elements of \( W_2 \) that are bigger than the least two elements of \( W_1 \). We now do this rigorously and more generally.

**Claim: For all \( e, R_e \) is satisfied:**

**Proof:** Fix \( e \). If \( W_e \) is finite then \( R_e \) is satisfied.

Assume \( W_e \) is infinite. We show that \( R_e \) is satisfied. Let \( x_1 < x_2 < \cdots < x_{2e} \) be the first (numerically) \( 2e \) elements of \( W_e \). Let \( s_0 \) be the least number such that

- \( x_1, \ldots, x_{2e} \in W_{e,s_0} \), and
- For all \( x \in \{ x_1, \ldots, x_{2e} \} \), for all \( 1 \leq i \leq e - 1 \), if \( x \in W_i \) then \( x \in W_{i,s_0} \).

**KEY:** for all \( s \geq s_0 \), during stage \( s \), the requirements \( R_1, \ldots, R_{e-1} \) may define \( \text{COL}(x, s) \) for some of the \( x \in \{ x_1, \ldots, x_{2e} \} \). But they will NOT define \( \text{COL}(x, s) \) for ALL of those \( x \). Why? Because \( R_i \) only defines \( \text{COL}(x, s) \) for at most TWO of those \( x \)’s, and there are \( e - 1 \) such \( i \), so at most \( 2e - 2 \) of those \( x \)’s have \( \text{COL}(x, s) \) defined. Hence there will exist \( x, y \) such that \( R_e \) gets to define \( \text{COL}(x, s) \) and \( \text{COL}(y, s) \). Furthermore, they will always be the SAME \( x, y \) since the \( R_i \) with \( i < e \) have already made up their minds about the \( x \) in \( \{ x_1, \ldots, x_{2e} \} \).

**UPSHOT:** There exists \( x, y \in W_e \) such that, for all \( s \geq s_0 \),

\( \text{COL}(x, s) = \text{RED} \)

\( \text{COL}(y, s) = \text{BLUE} \)

Since \( W_e \) is infinite there is SOME \( s \geq s_0 \) with \( s \in W_e \). Hence \( x, y, s \in W_e \) and show that \( W_e \) is NOT homogenous.

\[ \square \]
2 A Computable Coloring with NO c.e.-in-\(K\) Homog Sets

Notation 2.1

1. If \(A\) is a c.e. set, say \(A\) is the domain of \(M\), then \(A_s\) is \(\{x \leq s \mid M_{e,s}(x) \downarrow\}\). Note that, given \(s\), one can compute \(A_s\).

2. \(M_1^{(0)}, M_2^{(0)}, \ldots\) is a standard list of oracle Turing Machines.

3. Note that from \(e\) we can extract the code for \(M_e^{(0)}\).

4. If \(A\) is a c.e. set then \(M_{A_e}^{(s)}(x)\) means that we run \(M_e^{(0)}\) for \(s\) steps and using \(A_s\) for the oracle.

5. If \(A\) is c.e. then \(W_{e,A}\) is the domain of \(M_{e,A}^{(s)}\).

\[
W_{e,A}^{(s)} = \{x \mid (\exists s)[M_{e,s}^{A_e}(x) \downarrow]\}.
\]

Note that \(W_1^{K}, W_2^{K}, \ldots\) is a list of ALL c.e.-in-\(K\) sets.

6.

\[
W_{e,A}^{(s)} = \{x \mid M_{e,s}^{A_e}(x) \downarrow\}.
\]

Theorem 2.2 There exists \(COL : (\mathbb{N}^2) \rightarrow [2]\) such that there is NO infinite c.e.-in-\(K\) homog set.

Proof sketch: This will be a HW. But note that its very similar to the proof of Theorem 1.2— if \(W_e^{K}\) is infinite then eventually \(W_{e,s}^{K}\) will settle down on its first \(2e\) elements. □

3 A Computable Coloring with NO \(\Sigma_2\) Homog Sets

We state equivalences of both c.e. and c.e.-in-\(K\). We leave the proofs to the reader.

Theorem 3.1 Let \(A\) be a set. The following are equivalent:
1. There exists $e$ such that $A = W_e$. ($A$ is c.e.)

2. There exists a decidable $R$ such that

$$A = \{x \mid (\exists y)[(x, y) \in R]\}.$$  

($A$ is $\Sigma_1$.)

3. There exists $e$ such that

$$A = \{x \mid (\exists y, s)[M_{e,s}(y) = x]\}.$$  

(This is the origin of the phrase ‘computably ENUMERABLE.’)

**Theorem 3.2** Let $A$ be a set. The following are equivalent:

1. There exists $e$ such that $A = W^K_e$. ($A$ is c.e.-in-$K$.)

2. There exists a decidable-in-$K$ $R$ such that

$$A = \{x \mid (\exists y)[(x, y) \in R]\}.$$  

($A$ is $\Sigma^K_1$.)

3. There exists $e$ such that

$$A = \{x \mid (\exists y, s)[M^K_{e,s}(y) = x]\}.$$  

(This is the origin of the phrase ‘computably ENUMERABLE-in-$K$.’)

We also need to know that $K$ is quite powerful:

**Def 3.3** If $A, B$ are sets then $A \leq_m B$ means that there exists a computable $f$ such that

$$x \in A \iff f(x) \in B.$$
We leave the proof of the following to the reader.

**Theorem 3.4** If $A$ is c.e. then $A \leq_m K$.

The key use of the above theorem is that we can phrase $\Sigma_1$ questions as queries to $K$.

**Theorem 3.5** $A \in \Sigma_2 \iff A$ is c.e.-in-$K$.

**Proof:**

1) $A \in \Sigma_2$ implies $A$ is c.e.-in-$K$:

If $A \in \Sigma_2$ then there exists a TM $R$ that always converges such that

$$A = \{x \mid (\exists y)(\forall z)[R(x, y, z) = 1]\}.$$

Let $M^K$ be the TM that does the following:

1. Input$(x, y)$.

2. Ask $K(\forall z)[R(x, y, z) = 1]$. (Can rephrase as $(\exists z)[R(x, y, z) = 0].$)

3. If YES answer YES, if NO then answer NO.

$$A = \{x \mid (\exists y)[M^K(x, y) = 1]\}.$$

Hence $A$ is c.e.-in-$K$.

2) $A$ c.e.-in-$K$ implies $A \in \Sigma_2$.

$A$ is c.e.-in-$K$. So

$$A = W^K_{e} = \{x \mid (\exists s)(\forall t)[t \geq s \implies x \in W^K_{e,t}]\}.$$

So $A$ is $\Sigma_2$.  

\[\square\]
Theorem 3.6  There exists \( COL : \binom{N}{2} \to [2] \) such that there is NO infinite \( \Sigma_2 \) homog set.

Proof:  Combine Theorems 2.2 and 3.5. Note that we only need one part of the implication in Theorem 3.5.  ■

4  Every Computable Coloring has an Infinite \( \Pi_2 \) Homog set

We obtain this with a modification of the usual proof of Ramsey’s theorem. The key is that we don’t really toss things out - we guess on what the colors are and change our mind.

Theorem 4.1  For every computable coloring \( COL : \binom{N}{2} \to [2] \) there is an infinite \( \Pi_2 \) homog set.

Proof:

We are given computable \( COL : \binom{N}{2} \to [2] \).

CONSTRUCTION of \( x_1, x_2, \ldots \) and \( c_1, c_2, \ldots \).

NOTE: at the end of stage \( s \) we might have \( x_1, \ldots, x_i \) defined where \( i < s \). We will not try to keep track of how big \( i \) is. Also, we may have at stage (say) 1000 a sequence of length 50, and then at stage 1001 have a sequence of length only 25. The sequence will grow eventually but do so in fits and starts.

\begin{align*}
  x_1 & = 1 \\
  c_1 & = \text{RED} \text{  We are guessing. We might change our mind later}
\end{align*}

Let \( s \geq 2 \), and assume that \( x_1, \ldots, x_{s-1} \) and \( c_1, \ldots, c_{s-1} \) are defined.

1. Ask \( K \) Does there exists \( x \geq x_{s-1} \) such that, for all \( 1 \leq i \leq s - 1 \), \( COL(x_i, x) = c_i \)?
2. If YES then (using that \( \text{COL} \) is computable) find the least such \( x \). 

\[
x_i = x
\]

\( c_i = \text{RED} \) We are guessing. We might change our mind later

We have implicitly tossed out all of the numbers between \( x_{i-1} \) and \( x_i \).

3. If NO then we ask \( K \) how far back we can go. More rigorously we ask the following sequence of questions until we get a YES.

- Does there exists \( x \geq x_{s-1} \) such that, for all \( 1 \leq i \leq s - 2 \), \( \text{COL}(x_i, x) = c_i \)?
- Does there exists \( x \geq x_{s-1} \) such that, for all \( 1 \leq i \leq s - 3 \), \( \text{COL}(x_i, x) = c_i \)?
- \( \vdots \)
- Does there exists \( x \geq x_{s-1} \) such that, for all \( 1 \leq i \leq 2 \), \( \text{COL}(x_i, x) = c_i \)?
- Does there exists \( x \geq x_{s-1} \) such that, for all \( 1 \leq i \leq 1 \), \( \text{COL}(x_i, x) = c_i \)?

(One of these must be a YES since (1) if \( c_1 = \text{RED} \) and there are NO red edges coming out of \( x_1 \) then there must be an infinite number of \( \text{BLUE} \) edges, and (2) if \( c_1 = \text{BLUE} \) its because there are only a finite number of \( \text{RED} \) edges coming out of \( x_1 \) so there are an infinite number of \( \text{BLUE} \) edges. Let \( i_0 \) be such that There exists \( x \geq x_{s-1} \) such that, for all \( 1 \leq i \leq i_0 \), \( \text{COL}(x_i, x) = c_i \))

Do the following:

(a) Change the color of \( c_{i+1} \). (We will later see that this change must have been from \( \text{RED} \) to \( \text{BLUE} \).

(b) Wipe out \( x_{i+2}, \ldots, x_{s-1} \).

(c) Search for the \( x \geq x_{s-1} \) that the question asked says exist.

(d) \( x_{i+2} \) is now \( x \).
(e) $c_{i+2}$ is now RED.

END OF CONSTRUCTION of $x_1, x_2 \ldots$ and $c_1, c_2, \ldots$.

We need to show that there is a $\Pi_2$ homog set.

Let $X$ be the set of $x_i$ that are put on the board and stay on the board.

Let $R$ be the set of $x_i \in X$ whose final color is RED.

**Claim 1:** Once a number turns from RED to BLUE it can’t go back to RED again.

**Proof:**

If a number is turned BLUE its because there are only a finite number of RED edges coming out of it. Hence there must be an infinite number of BLUE edges coming out of it. Hence it will never change color (though it may be tossed out).

**End of Proof**

**Claim 1:** $X, R \in \Pi_2$.

**Proof:**

We show that $X \in \Sigma_2$. In order to NOT be in $X$ you must have, at some point in the construction, been tossed out.

$$X = \{ x \mid (\exists x)[\text{at stage } s \text{ of the construction } x \text{ was tossed out}] \}.$$  

Note that the condition is computable-in-$K$. Hence $X$ is c.e.-in-$K$. By Theorem 3.5 $X \in \Sigma_2$.

We show that $R \in \Sigma_2$. In order to NOT be in $R$ you must have to either NOT be in $X$ or have been turned blue. Note that once you turn at some point in the construction, been tossed out.

$$R = \overline{X} \cup \{ x \mid (\exists x)[\text{at stage } s \text{ of the construction } x \text{ was turned BLUE}] \}.$$  

Recall that $\Sigma_2$ is closed under complementation. So we only need to show that the other union-and is in $\Sigma_2$. Note that the condition is computable-in-$K$. Hence $\overline{R}$ is c.e.-in-$K$. By Theorem 3.5
$R \in \Sigma_2$.

**End of Proof**

There are two cases:

1. If $R$ is infinite then $R$ is an infinite homog set that is $\Pi_2$.

2. If $R$ is finite then $B$ is $X$ minus a finite number of elements. Since $X$ is $\Pi_2$, $B$ is $\Pi_2$.

References