Public Key Cryptography: Attacks on RSA, NON-RSA Encryption
Public Key Cryptography: Low $e$ Attacks on RSA
Needed Math: Chinese Remainder Theorem Example

Find $x$ such that:

\[
\begin{align*}
x &\equiv 17 \pmod{31} \\
x &\equiv 20 \pmod{37}
\end{align*}
\]

a) The inverse of $31 \mod 37$ is 6
b) The inverse of $37 \mod 31$ is the inverse of $6 \mod 31$ which is 26.
c) $20 \times 6 \times 31 + 17 \times 26 \times 37 = 20,074$

\[
20 \times (31)^{-1} \times 31 + 17 \times (37)^{-1} \times 37
\]

Mod 31: First term is 0. Second term is 17. So 17.
Mod 37: First term is 20. Second term is 0. So 20.
So $x = 20,074$ is answer.
Needed Math: Chinese Remainder Theorem Example

Find \( x \) such that:

\[
x \equiv 17 \pmod{31} \quad \& \quad x \equiv 20 \pmod{37}
\]

So \( x = 20,074 \) is answer. Can we find a smaller \( x \)?
We only care about \( x \pmod{31} \) and \( x \pmod{37} \).

Note:

\[
x \equiv 17 \pmod{31} \quad \implies x - 31 \times 37 \equiv 17 \pmod{31}
\]

\[
x \equiv 20 \pmod{37} \quad \implies x - 31 \times 37 \equiv 20 \pmod{37}
\]

If \( x \) works then \( x - 31 \times 37 \) works. Iterate until get between 0 and \( 31 \times 37 \). What's this called? **Discuss**
Find $x$ such that:

$$x \equiv 17 \pmod{31} \quad \& \quad x \equiv 20 \pmod{37}$$

So $x = 20,074$ is answer. Can we find a smaller $x$? We only care about $x \pmod{31}$ and $x \pmod{37}$.

**Note:**

$$x \equiv 17 \pmod{31} \implies x - 31 \times 37 \equiv 17 \pmod{31}$$

$$x \equiv 20 \pmod{37} \implies x - 31 \times 37 \equiv 20 \pmod{37}$$

If $x$ works then $x - 31 \times 37$ works. Iterate until get between 0 and $31 \times 37$. What's this called? **Discuss $x \pmod{31 \times 37}$**
Find $x$ such that:

$$x \equiv 17 \pmod{31} \quad \& \quad x \equiv 20 \pmod{37}$$

So $x = 20,074$ is answer. Can we find a smaller $x$?

We only care about $x \pmod{31}$ and $x \pmod{37}$.

**Note:**

$$x \equiv 17 \pmod{31} \implies x - 31 \times 37 \equiv 17 \pmod{31}$$

$$x \equiv 20 \pmod{37} \implies x - 31 \times 37 \equiv 20 \pmod{37}$$

If $x$ works then $x - 31 \times 37$ works. Iterate until get between 0 and $31 \times 37$. What's this called? **Discuss** $x \pmod{31 \times 37}$

**Upshot:** Can take $x = 20,074 \pmod{31 \times 37} = 629$
Needed Math: Chinese Remainder Theorem \( L = 2 \) Case

1. Input \( a, b, N_1, N_2, N_1, N_2, \) rel primes. Want \( 0 \leq x \leq N_1 N_2 \):

\[
x \equiv a \pmod{N_1} \\
x \equiv b \pmod{N_2}
\]

2. Find the inverse of \( N_1 \) mod \( N_2 \) and denote this \( N_1^{-1} \).

3. Find the inverse of \( N_2 \) mod \( N_1 \) and denote this \( N_2^{-1} \).

4. \( y = b N_1^{-1} p + a N_2^{-1} q \)
   
   Mod \( N_1 \): 1st term is 0, 2nd term is \( a \). So \( y \equiv a \pmod{N_1} \).
   
   Mod \( N_2 \): 2nd term is 0, 1st term is \( b \). So \( y \equiv b \pmod{N_2} \).

5. \( x \equiv y \pmod{N_1 N_2} \). (Convention that \( 0 \leq x \leq N_1 N_2 - 1 \))
Theorem: If $N_1, \ldots, N_L$ are rel prime, $x_1, \ldots, x_L$ are anything, then there exists $x$ with $0 \leq x \leq N_1 \cdots N_L$ such that

\[ x \equiv x_1 \pmod{N_1} \]
\[ x \equiv x_2 \pmod{N_2} \]
\[ \vdots \]
\[ x \equiv x_L \pmod{N_L} \]

Proof: On HW.

Notation: CRT is Chinese Remainder Theorem.
Theorem: Assume $N_1, N_2$ are rel prime, $e, m \in \mathbb{N}$. Assume there is an $x$ (NOT necc. $\leq N_1 N_2$) such that
\[ x \equiv m^e \pmod{N_1} \]
\[ x \equiv m^e \pmod{N_2} \]
Then $x \equiv m^e \pmod{N_1 N_2}$.

Proof: There exists $k_1, k_2$ such that
\[ x = m^e + k_1 N_1 \]
\[ x = m^e + k_2 N_2 \]
Subtract to get $k_1 N_1 = k_2 N_2$. Since $N_1, N_2$ rel prime, $N_1$ divides $k_2$, so $k_2 = k N_1$.
\[ x = m^e + k N_1 N_2. \] Hence $x \equiv m^e \pmod{N_1 N_2}$. 

Needed Math: The e Theorem, $L = 2$ case
Theorem: Assume $N_1, \ldots, N_L$ are rel prime, $e, m \in \mathbb{N}$. Assume there is an $x$ (NOT necc $\leq N_1 \cdots N_L$) such that

\[
x \equiv m^e \pmod{N_1} \\
\vdots \\
x \equiv m^e \pmod{N_L}
\]

Then $x \equiv m^e \pmod{N_1 \cdots N_L}$.

Proof: Might be on a future HW, or Midterm, or Final, or any combination of the three. Or might not.
Low Exponent Attack: Example

1) $N_a = 377$, $N_b = 391$, $N_c = 589$. For Alice, Bob, Carol.
2) $e = 3$.
3) Zelda sends $m$ to all three. Eve will find $m$. Note $m < 377$.

1. Zelda sends Alice 330. So $m^3 \equiv 330 \pmod{377}$.
2. Zelda sends Bob 34. So $m^3 \equiv 34 \pmod{391}$.
3. Zelda sends Carol 419. So $m^3 \equiv 419 \pmod{589}$.

Eve sees all of this. Eve uses CRT to find $0 \leq x < 377 \times 391 \times 589$.

$x \equiv 330 \equiv m^3 \pmod{377}$
$x \equiv 34 \equiv m^3 \pmod{391}$
$x \equiv 419 \equiv m^3 \pmod{589}$

Eve finds such a number: $x = 1,061,208$.

By e-Theorem

$1,061,208 \equiv m^3 \pmod{377 \times 391 \times 589}$. 
By $e$-Theorem

$$1,061,208 \equiv m^3 \pmod{377 \times 391 \times 589}.$$  

**Most Important Fact:** Recall that $m \leq 377$. Hence note that:

$$m^3 < 377 \times 391 \times 589$$

$$m^3 \equiv 1,061,208 \pmod{377 \times 391 \times 589}$$

Therefore the $m^3$ calculation cannot have wrap-around. Hence $m$ can be gotten from the ordinary cube root operation. We find

$$(1,061,208)^{1/3} = 102$$

So $m = 102$,

**Note:** Cracked RSA without factoring.
Low Exponent Attack: Generalized

1) $L$ people. Use $N_1 < \cdots < N_L$. All Rel Prime.
2) $e \leq L$
3) Zelda sends $m$ to $L$ people. Note $m < N_1$. 

You will finish this on HW. You will write psuedocode.
Can you run the algorithm even if $e$ is not small? Discuss Yes- and if $m$ is small enough it may even work. But it needs to report FAILURE if get $x > N_1 \cdots N_L$. 

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Can you run the algorithm even if $e$ is not small? **Discuss**
Yes- and if $m$ is small enough it may even work. But it needs to report FAILURE if get $x > N_1 \cdots N_L$. 
Public Key Cryptography: NON-RSA Encryption
RSA

Let $n$ be a security parameter

1. Alice picks two primes $p, q$ of length $n$ and computes $N = pq$.
2. Alice computes $\phi(N) = \phi(pq) = (p-1)(q-1)$. Denote by $R$
3. Alice picks an $e \in \{ \frac{R}{3}, \ldots, \frac{2R}{3} \}$ that is relatively prime to $R$. Alice finds $d$ such that $ed \equiv 1 \pmod{R}$.
4. Alice broadcasts $(N, e)$. (Bob and Eve both see it.)
5. Bob: To send $m \in \{1, \ldots, N-1\}$, send $m^e \pmod{N}$.
6. If Alice gets $m^e \pmod{N}$ she computes

   $$ (m^e)^d \equiv m^{ed} \equiv m^d \equiv 1 \pmod{R} \equiv m $$
Is RSA Hard to Crack?

**Hardness Assumption for RSA:** The following problem is hard:
Given \((N, e, c)\) where \(N = pq\) and \(c \equiv m^e \pmod{N}\) for some \(m\),
Find \(m\).

**Objection:** Hardness assumption not natural.
**Objection:** Hardness assumption does not have a long history of being tested.
**We Want:** An Encryption scheme based on Factoring being hard.

Is there one? **Vote:** Yes, No, or Unk?
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We Want: An Encryption scheme based on Factoring being hard.

Is there one? Vote: Yes, No, or Unk?
Yes. Rabin Encryption.
Rabin Encryption
1. Solve \( m^2 \equiv 1 \pmod{7} \)
Math for Rabin Encryption – Square Roots Mod 7

1. Solve $m^2 \equiv 1 \pmod{7}$ $m = 1, 6$
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2. Solve \( m^2 \equiv 2 \) (mod 7)
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2. Solve $m^2 \equiv 2 \pmod{7}$ $m = 3, 4$
3. Solve $m^2 \equiv 3 \pmod{7}$

Since $a^2 = (-a)^2$ will always have, for all prime $p$, $p-1$ elements of \{1, ..., $p$\} have sqrts mod $p$.

$p-1$ elements of \{1, ..., $p$\} do not have sqrts mod $p$.

Note: Computing Square Roots Mod n will mean determining if they exists and if so return all of them.
Math for Rabin Encryption – Square Roots Mod 7

1. Solve \( m^2 \equiv 1 \pmod{7} \) \( m = 1, 6 \)
2. Solve \( m^2 \equiv 2 \pmod{7} \) \( m = 3, 4 \)
3. Solve \( m^2 \equiv 3 \pmod{7} \) NONE

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Note: Computing Square Roots Mod n will mean determining if they exist and if so return all of them.
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4. Solve \( m^2 \equiv 4 \pmod{7} \) \( m = 2, 5 \)
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Since \( a^2 = (-a)^2 \) will always have, for all prime \( p \), \( \frac{p-1}{2} \) elements of \( \{1, \ldots, p\} \) have sqrts mod \( p \). \( \frac{p-1}{2} \) elements of \( \{1, \ldots, p\} \) do not have sqrts mod \( p \).

Note: Computing Square Roots Mod \( n \) will mean determining if they exists and if so return all of them.
Theorem: \( c \) has a sqrt mod \( p \) iff \( c^{(p-1)/2} - 1 \equiv 0 \).

\[
c = m^2 \implies c^{(p-1)/2} \equiv (m^2)^{(p-1)/2} \equiv m^{p-1} \equiv 1.
\]

The equation \( x^{(p-1)/2} - 1 \equiv 0 \) has \( (p-1)/2 \) roots.
There are \( (p-1)/2 \) numbers that have sqrts. Hence
If \( c \) does not have a sqrt root then \( c^{(p-1)/2} - 1 \not\equiv 0 \).

Theorem: If \( p \equiv 3 \pmod{4} \) then easy to compute sqrt mod \( p \).
Given \( c \) if \( c^{(p-1)/2} \not\equiv 1 \) NO. If \( \equiv 1 \) then:

\[
(c^{(p+1)/4})^2 \equiv c^{(p+1)/2} \equiv c(c^{(p-1)/2}) \equiv c \times 1 \equiv c.
\]

So output \( c^{(p+1)/4} \) and other sqrt is \( p - c^{(p+1)/4} \).

Note: If \( p \equiv 1 \pmod{4} \) also easy to do sqrt.

Upshot: Sqrt mod a prime is easy!
What about sqrt mod a composite. Try these:

1. Solve \( m^2 \equiv 9 \pmod{1147} \)
2. Solve \( m^2 \equiv 101 \pmod{1147} \)
Math for Rabin Encryption – Square Roots Mod \( n \)

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   - Answers: 3, 34, 1113, 1144.
2. Solve \( m^2 \equiv 101 \) (mod 1147): Hmmm.

Vote: Is finding sqrts mod \( N \) hard? Yes, No, Unk?
Unk: Many computational questions in Number Theory are Unk.
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Solve \( m^2 \equiv 9 \pmod{1147} \): 3, \( 1147 - 3 = 1144 \) easy. If had 34 then \( 1147 - 34 = 1144 \) easy. But how to get 34?

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\[ m^2 \equiv 101 \pmod{1147} \quad 1147 = 31 \times 37 \]

\[ m^2 \equiv 101 \pmod{31}. \quad m^2 \equiv 8 \pmod{31}: \quad m \equiv \pm 15 \pmod{31} \]
\[ m^2 \equiv 101 \pmod{37}. \quad m^2 \equiv 27 \pmod{37} \quad m \equiv \pm 8 \pmod{37}. \]

One approach: Want number \( m \in \{1, \ldots, 1146\} \) such that \( m \equiv 15 \pmod{31} \)
\( m \equiv 8 \pmod{37} \)

\[ m = 15x + 8y \]

Use CRT to get:

\[ m = 15918 \equiv 1007 \pmod{1147} \]
By using $\pm 15 \pmod{31}$ and $\pm 8 \pmod{37}$ can find 4 sqrts.

**Upshot:** sqrts mod $N$ easy if know the factors of $n$.

**Upshot:** Always get 0 or 2 or 4 sqrts if mod $N = pq$.

What about finding sqrts mod $N$ where factors of $N$ are not known?
Math for Rabin Encryption – Square Roots Mod \( n \)

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What about finding sqrts mod \( N \) where factors of \( N \) are not known?

Normally I would say

The problem of finding sqrt mod \( N \) where the factors of \( N \) are not known is believed to be hard.
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This time I can say something stronger.
Math for Rabin Encryption – Square Roots Mod \( n \)

How hard is sqrts mod \( N \) when factors of \( N \) not known?

Theorem: If finding sqrts mod \( N \) is easy then factoring is easy.

1. Given \( N = pq \) \((p, q \text{ unknown})\) want to factor it.
2. Pick a random \( c \) and find its sqrts.
3. If it doesn't have \( \geq 4 \) sqrts then goto step 2.
4. The four sqrts are of the form \( \pm x \) and \( \pm y \). Now use \( x \), \( y \). We know that \( x^2 \equiv y^2 \pmod{N} \).
   \[ x^2 - y^2 \equiv 0 \pmod{N} \]
   \( (x - y)(x + y) \equiv 0 \pmod{N} \)
   \( \text{GCD} (x - y, N) \) or \( \text{GCD} (x + y, N) \) likely factor.

Discuss: Why did I use \( x \), \( y \) instead of \( x \), \( -x \)?
Math for Rabin Encryption – Square Roots Mod $n$

How hard is sqrts mod $N$ when factors of $N$ not known?

**Theorem:** If finding sqrts mod $N$ is easy then factoring is easy.

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3. If it doesn’t have $\geq 4$ sqrts then goto step 2.
4. The four sqrts are of the form $\pm x$ and $\pm y$. Now use $x, y$. We know that

\[ x^2 \equiv y^2 \pmod{N}. \]

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$GCD(x - y, N)$ or $GCD(x + y, N)$ likely factor.

**Discuss:** Why did I use $x, y$ instead of $x, -x$?
All you Need to Know for Rabin’s Scheme

1. Finding primes is easy.
2. Squaring is easy.
3. If $N$ is factored then $\sqrt{N}$ mod $N$ is easy.
4. If $N$ is not factored then $\sqrt{N}$ mod $N$ is thought to be hard (equiv to factoring).
Rabin’s Encryption Scheme

$n$ is a security parameter

1. Alice gen $p, q$ primes of length $n$. Let $N = pq$. Send $N$.
2. **Encode:** To send $m$, Bob sends $c = m^2 \pmod{N}$.
3. **Decode:** Alice can find $m$ such that $m^2 \equiv c \pmod{N}$.
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**PRO:** Easy for Alice and Bob

**BIG PRO:** Factoring Hard is hardness assumption.

**CON:** Alice has to figure out which of the sqrts is correct message.

**Caveat:** If $m$ is English text then Alice can tell which one it is.

**Caveat:** If not. Hmmm.
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How to Modify Rabin’s Encryption?

Let’s look at mod $21 = 3 \times 7$.

- $1^2, 8^2, 13^2, 20^2 \equiv 1$
- $2^2, 5^2, 16^2, 19^2 \equiv 4$
- $3^2, 18^2 \equiv 9$
- $4^2, 10^2, 11^2, 17^2 \equiv 16$
- $6^2, 15^2 \equiv 15$
- $7^2, 14^2 \equiv 7$
- $9^2, 12^2 \equiv 18$

**Question:** What do the red numbers have in common? Discuss
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**Question:** What do the red numbers have in common? **Discuss**

They all have square roots! They are all also on the RHS.
How to Modify Rabin’s Encryption?

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$2^2, 5^2, 16^2, 19^2 \equiv 4$
$3^2, 18^2 \equiv 9$
$4^2, 10^2, 11^2, 17^2 \equiv 16$
$6^2, 15^2 \equiv 15$
$7^2, 14^2 \equiv 7$
$9^2, 12^2 \equiv 18$

**Question:** What do the red numbers have in common? Discuss.

They all have square roots! They are all also on the RHS.
What is it about 21 that makes this work?
Definition: A Blum Int is product of two primes \( \equiv 3 \pmod{4} \).

Example: \( 21 = 3 \times 7 \).

Notation: \( SQ_N \) is the set of squares mod \( N \). (Often called \( QR_N \).)

Example: If \( N = 21 \) then \( SQ_N = \{ 1, 4, 7, 9, 15, 16, 18 \} \).

Theorem: Assume \( N \) is a Blum Integer. Let \( m \in SQ_N \). Then of the two or four sqrts of \( m \), only one is itself in \( SQ_N \).

Proof: Omitted. Note: (1) not that hard, and (2) in Katz book.

We use Theorem to modify Rabin Encryption.
Rabin’s Encryption Scheme 2.0

(This modification by Blum and Williams.) $n$ is a security parameter.

1. Alice gen $p, q$ primes of length $n$ such that $p, q \equiv 3 \pmod{4}$. Let $N = pq$. Send $N$.
2. Encode: To send $m$, Bob sends $c = m^2 \pmod{N}$. Only send $m$’s in $SQ_N$.
3. Decode: Alice can find 2 or 4 $m$ such that $m^2 \equiv c \pmod{N}$. Take the $m \in SQ_N$.

**PRO:** Easy for Alice and Bob

**Biggest PRO:** Factoring Hard is hardness assumption.

**CON:** Messages have to be in $SQ_N$. 
Can Rabin’s Encryption Scheme Can Be Cracked?

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2. Encode: To send $m$, Bob sends $c = m^2 \pmod{N}$.
3. Decode: Alice can find $m$ such that $m^2 \equiv c \pmod{N}$. Picks a poss out somehow.

Vote: Crackable, Uncrackable, Unk
Can Rabin’s Encryption Scheme Can Be Cracked?

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Crackable:

Attack!: Eve picks an $m$ and tricks Alice into sending message $m$ via $m^2 \equiv c$. Eve is hoping that Bob will find another sqrt of $m^2$. Say Alice gets $m'$. Then $m^2 - (m')^2 \equiv 0 \pmod{N}$. $(m - m')(m + m') \equiv 0 \pmod{N}$. $m - m'$ or $m + m'$ may share factors with $N$ so do $gcd(m - m', N)$ and $gcd(m + m', N)$. Can factor $N$ and hence – game over!
What else to known

1. Alice may need to guess which of the 2 or 4 possible messages is the one to use, which is why it's not used. Blum and Williams showed how to make the message unique, but by the time they did RSA was pervasive.

2. RSA and Rabin have similar issues which require padding randomness.

3. RSA has also had attacks as we've seen.

4. Rabin can be cracked with Chosen Plaintext Attack.

5. There is a variant of Rabin that thwarts the CPA but not provably equiv to factoring.

Alternate History: Had timing been different Rabin would have been the one everyone uses.