MIDTERM REVIEW-ADMIN
1) Midterm is Monday Oct 29 2-3:15 in class
2) Can bring one sheet of notes.
   Can use both sides
   Can be typed
   You can put whatever you want on it.
   Can copy a classmates and use it but thats stupid
   Can try to cram the entire course onto it but thats stupid
3) No calculators allowed. Numbers will be small.
4) Coverage: Slides/HW.
5) Not on Exam: Guest lectures.
6) We hope to grade it and post it Monday Night.
7) If can’t take the exam tell me ASAP.
8) Advice: Understand rather than memorize.
MIDTERM REVIEW-CONTENT
Alice, Bob, and Eve

- Alice sends a message to Bob in code.
- Eve overhears it.
- We want Eve to not get any information.

There are many aspects to this:
- Information-Theoretic Security.
- Computational-Theoretic Security (Hardness Assumption)
- The NY, NY problem: Do not always code $m$ the same way.
- Private Key or Public key
- Kerckhoff’s principle: Eve knows cryptosystem.
- History: How much computing power does Eve have?
The First Steps in Any Cipher

1. Get rid of spacing
2. Get rid of punctuation
3. Make everything capital letters
4. Class convention: we use either
   4.1 $\Sigma = \{A, \ldots, Z\}$, or
   4.2 $\Sigma = \{0, \ldots, n-1\}$ ($n$ is often a prime), or
   4.3 $\Sigma = \{0, 1\}$. 
Private Key Ciphers
1. Shift cipher: \( f(x) = x + s. \) \( s \in \{0, \ldots, 25\}. \)

2. Affine cipher: \( f(x) = ax + b. \) \( a, b \in \{0, \ldots, 25\}. \) \( a \) rel prime 26.

3. Keyword Shift: From keyword and shift create random-looking perm of \( \{a, \ldots, z\} \).

4. Keyword Mixed: From keyword create random-looking perm of \( \{a, \ldots, z\} \).

5. Gen Sub Cipher: Take random perm of \( \{a, \ldots, z\} \).
Important: Algorithm Is-English.

1. Input($T$) a text
2. Find $f_T$, the freq vector of $T$
3. Find $x = f_T \cdot f_E$ where $f_E$ is freq vector for English
4. If $x \geq 0.06$ then output YES. If $x \leq 0.04$ then output NO. If $0.04 < x < 0.06$ then something is wrong.

1. Shift , Affine have small key space: can try all keys and see when Is-English says YES.
2. For others use Freq analysis, e.g., e is most common letter.
3. If message is numbers (e.g., Credit Cards) or ASCII (e.g., Byte-Shift) there are still patterns so can use freq analysis.
Letter Frequencies
Randomized Shift

How to NOT encode the same $m$ the same way:

**Randomized shift:** Key is a function $f : S \to S$.

1. To send message $(m_1, \ldots, m_L)$ (each $m_i$ is a character)
   1.1 Pick random $r_1, \ldots, r_L \in S$. For $1 \leq i \leq L$ compute $s_i = f(r_i)$.
   1.2 Send $((r_1; m_1 + s_1), \ldots, (r_L; m_L + s_L))$

2. To decode message $((r_1; c_1), \ldots, (r_L; c_L))$
   2.1 For $1 \leq i \leq L$ $s_i = f(r_i)$.
   2.2 Find $(c_1 - s_1, \ldots, c_L - s_L)$

**Note:** Can be cracked.
The key is $f(r) = 2r + 7$. Alice wants to send NY,NY which we interpret as nyny. Need four shifts.

Pick random $r = 4$, so first shift is $2 \times 4 + 7 = 15$
Pick random $r = 10$, so second shift is $2 \times 10 + 7 = 1$
Pick random $r = 1$, so third shift is $2 \times 1 + 7 = 9$
Pick random $r = 17$, so fourth shift is $2 \times 17 + 7 = 15$

Send (4;C), (10,Z), (1,W), (17,N)

Eve will not be able to tell that is of the form XYXY.
The Vigenère cipher

Key: \( k = (k_1, k_2, \ldots, k_n) \).

Encrypt (all arithmetic is mod 26)

\[
Enc(m_1, m_2, \ldots, m_N) = m_1 + k_1, m_2 + k_2, \ldots, m_n + k_n, m_{n+1} + k_1, m_{n+2} + k_2, \ldots, m_{n+n} + k_n, \ldots
\]

Decrypt Decryption just reverse the process
1. Find Keylength or set $K$ of them. Either try length 1, 2, 3, ... or find repeated strings of letters so can guess.

2. Let $K$ be the set of possible key lengths. For every $L \in K$:
   2.1 Separate text $T$ into $L$ streams depending on position mod $L$
   2.2 For each steam try every shift and use Is-English to determine which shift is correct.
   2.3 You now know all shifts for all positions. Decrypt!
Getting More Out of Your Phrase

If the key was Corn Flake key of length 9. Want More.
We form a key of length \( LCM(4, 5) = 20 \).

\[
\begin{array}{cccccccccc}
C & O & R & N & C & O & R & N & C & O \\
F & L & A & K & E & F & L & A & K & E \\
\hline
7 & 25 & 17 & 23 & 6 & 19 & 2 & 13 & 12 & 18 \\
\end{array}
\]

ADD it up to get new 20-long key.

Wheel of Fortune yields key size \( LCM(5, 2, 7) = 70 \).
Vig Book Cipher

Use Book for key.

1. LONG key- great!
2. Should pick obscure book (see next slide).
3. Crackable NOW by looking at common pairs-of-letters since both book and message are English.
4. Probably hard in 1776.
Ever notice how civilians (that is non-math people) use math words badly? Ever notice how sometimes you know a math statement is false (or not known) since if it was true you would know it?

Each chapter of this book makes a point like those above and then illustrates the point by doing some real mathematics.

This book gives readers valuable information about how mathematics and theoretical computer science work, while teaching them some actual mathematics and computer science through examples and exercises. Most of the mathematics could be understood by a bright high school student. The points made can be understood by anyone with an interest in math, from the bright high school student to a Field’s medal winner.
1. Shift
2. Affine
3. Keyword Shift
4. Keyword Mixed
5. Gen Sub
6. Vig
7. all 1-letter substitutions.

Freq cracked them (for Vig Freq plus some other stuff).

Idea: Sub $n$ letters at a time.

Need bijection of $\{0, \ldots, 25\}^n$ to $\{0, \ldots, 25\}^n$ that is easy to use.
The Matrix Cipher

**Definition:** Matrix Cipher. Pick $n$ and $M$ an $n \times n$ invertible matrix.

1. Encrypt via $\vec{x} \rightarrow M(\vec{x})$.
2. Decrypt via $\vec{y} \rightarrow M^{-1}(\vec{y})$

We’ll take $n = 30$. 

1. Easy for Alice and Bob.
2. Key $M$ is small enough to be easy for Alice and Bob but too large for Eve to use brute force.
3. Eve can crack using freqs of 30-long sets of letters? Hard?
4. Ciphertext only might be uncrackable.
5. Can crack from message-cipher pairs.
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If $n \times n$ matrix then keyspace has roughly $26^{n^2}$.

1. Trying every matrix takes $26^{n^2}$.
2. If guess one row at a time then $O(n26^n)$.
3. Lesson: Eve may think of attacks you had not thought of.
4. Lesson: Attacks can be thwarted, once known, by increasing $n$
One-time pad

1. Let $\mathcal{M} = \{0, 1\}^n$

2. $\text{Gen}$: choose a uniform key $k \in \{0, 1\}^n$

3. $\text{Enc}_k(m) = k \oplus m$

4. $\text{Dec}_k(c) = k \oplus c$

5. Correctness:

$$\text{Dec}_k(\text{Enc}_k(m)) = k \oplus (k \oplus m)$$
$$= (k \oplus k) \oplus m$$
$$= m$$
PROS AND CONS Of One-time pad

1. If Key is $N$ bits long can only send $N$ bits.
2. $\oplus$ is FAST!
3. The one-time pad is uncrackable. YEAH!
4. Generating truly random bits is hard. BOO!
5. Psuedo-random can be insecure – I did example.
Public Key Ciphers
Eve can go . . .
Public Key Cryptography

Alice and Bob never have to meet!
All arithmetic is mod $p$. The following can be done quickly.

1. Given $(a, n, p)$ compute $a^n \pmod{p}$. Repeated Squaring. (1) $\leq 2 \lg n$ always, (2) $\leq \lg n + O(1)$ if $n$ close to $2^{2^m}$.
2. Given $n$, find a safe prime of length $n$ and a generator $g$.
3. Given $a, b$ rel prime find inverse of $a \pmod b$: Euclidean alg.
4. Given $a_1, \ldots, a_L$ and $b_1, \ldots, b_L$, $b_i$’s rel prime, find $x \equiv a_i \pmod{b_i}$.
5. Given $(a, p)$ find $\sqrt{a}$’s. We did $p \equiv 3 \pmod{4}$ case.
6. Given $(a, N)$ and $p, q$ such that $N = pq$, find $\sqrt{a}$’s.
Number Theory Assumptions

1. Discrete Log is hard.
2. Factoring is hard.
3. Given \((a, N)\), find \(\sqrt{a}\) without being given factors of \(N\) is hard. (This is equiv to factoring.)

Note: We usually don’t assume these but instead assume close cousins.
The Diffie-Helman Key Exchange

Alice and Bob will share a secret $s$.

1. Alice finds a $(p, g)$, $p$ of length $n$, $g$ gen for $\mathbb{Z}_p$. Arith mod $p$.
2. Alice sends $(p, g)$ to Bob in the clear (Eve can see it).
3. Alice picks random $a \in \{1, \ldots, p - 1\}$. Alice computes $g^a$ and sends it to Bob in the clear (Eve can see it).
4. Bob picks random $b \in \{1, \ldots, p - 1\}$. Bob computes $g^b$ and sends it to Alice in the clear (Eve can see it).
5. Alice computes $(g^b)^a = g^{ab}$.
6. Bob computes $(g^a)^b = g^{ab}$.
7. $g^{ab}$ is the shared secret.
The Diffie-Helman Key Exchange

Alice and Bob will share a secret \( s \).

1. Alice finds a \((p, g)\), \( p \) of length \( n \), \( g \) gen for \( \mathbb{Z}_p \). Arith mod \( p \).
2. Alice sends \((p, g)\) to Bob in the clear (Eve can see it).
3. Alice picks random \( a \in \{1, \ldots, p-1\} \). Alice computes \( g^a \) and sends it to Bob in the clear (Eve can see it).
4. Bob picks random \( b \in \{1, \ldots, p-1\} \). Bob computes \( g^b \) and sends it to Alice in the clear (Eve can see it).
5. Alice computes \((g^b)^a = g^{ab}\).
6. Bob computes \((g^a)^b = g^{ab}\).
7. \( g^{ab} \) is the shared secret.

Definition
Let \( f \) be \( f(p, g, g^a, g^b) = g^{ab} \).

Hardness assumption: \( f \) is hard to compute.
How Useful is Diffie-Helman

**CAVEAT:** DH is not a cipher.

**PRO:** Alice and Bob can use $g^{ab}$ to transmit a key for a cipher.

**Used?** DH is at used in many real authentication schemes!
ElGamal is DH with a Twist

1. Alice and Bob do Diffie Helman.
2. Alice and Bob share secret $s = g^{ab}$.
3. Alice and Bob compute $(g^{ab})^{-1} \pmod{p}$.
4. To send $m$, Alice sends $c = mg^{ab}$
5. To decrypt, Bob computes $c(g^{ab})^{-1} \equiv mg^{ab}(g^{ab})^{-1} \equiv m$

We omit discussion of Hardness assumption (HW)
Needed Math for RSA – The $\phi$ Function

Definition

$\phi(n)$ is the numb of nums in $\{1, \ldots, n - 1\}$ that are rel prime to $n$.

Note: If $p$ is prime then $\phi(p) = p - 1$.

Known: If $n$ is any number then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Ramifications: For all $m$, $a^m \equiv a^m \pmod{\phi(n)} \pmod{n}$.

Known: If $a, b$ are relatively prime then $\phi(ab) = \phi(a)\phi(b)$.

Known: Given $R$, easy to find $e$ rel prime to $R$ and $d$ such that $ed \equiv 1 \pmod{R}$.

Believe: Let $N = pq$, $R = (p - 1)(q - 1)$ and $e$ rel prime to $R$. If know $N$ but Not $R$ then hard to find $d$ with $ed \equiv 1 \pmod{R}$. 
RSA

Let $n$ be a security parameter

1. Alice picks two primes $p, q$ of length $n$ and computes $N = pq$.
2. Alice computes $\phi(N) = \phi(pq) = (p - 1)(q - 1)$. Denote by $R$
3. Alice picks an $e \in \left\{ \frac{R}{3}, \ldots, \frac{2R}{3} \right\}$ that is relatively prime to $R$.
   Alice finds $d$ such that $ed \equiv 1 \pmod{R}$.
4. Alice broadcasts $(N, e)$. (Bob and Eve both see it.)
5. Bob: To send $m \in \{1, \ldots, N - 1\}$, send $m^e \pmod{N}$.
6. If Alice gets $m^e \pmod{N}$ she computes

\[ (m^e)^d \equiv m^{ed} \equiv m^{ed} \pmod{R} \equiv m^1 \pmod{R} \equiv m \]
Hardness Assumption for RSA

Definition: Let $f$ be $f(N, e) = d$, where $N = pq$, and

$$ed \equiv 1 \pmod{(p - 1)(q - 1)}$$

Hardness assumption (HA): $f$ is hard to compute.
Plain RSA Bytes!

The RSA given above is referred to as Plain RSA. **Insecure!**  $m$ is always coded as $m^e \pmod{N}$.

Make secure by padding:  $m \in \{0, 1\}^{L_1}$,  $r \in \{0, 1\}^{L_2}$.

To send  $m \in \{0, 1\}^{L_1}$, pick rand  $r \in \{0, 1\}^{L_2}$, send  $(rm)^e$.

(NOTE-  $rm$ means $r$ CONCAT with $m$ here and elsewhere.)

**DEC:** Alice finds $rm$ and takes leftmost $L_1$ bits.

**Caveat:** RSA still has issues when used in real world. They have been fixed. Maybe.
Attacks on RSA

1. Factoring Algorithms. We saw some ideas with Jevon’s Number. **Response:** Pick larger $p, q$

2. If Zelda give $A_i \ (N_i, e)$:
   2.1 Low-$e$ attack: **Response:** High $e$. Duh.
   2.2 $m^e < N_1 \cdots N_L$: **Response:** Pad $m$.

3. If Zelda give $A_i \ (N, e_i)$ and two of the $e_i$’s are rel prime, then Euclidean Alg Attack: **Response:** Give everyone diff $N$’s. Duh.

4. Timing Attacks: **Response:** Pad the amount of time used.

**Caveat:** Theory says use different $e$’s. Practice says use $e = 2^{16} + 1$ for speed.
Math for Rabin Encryption – Procedures

How to find square roots mod $p$ if $p \equiv 3 \pmod{4}$. All arithmetic is mod $p$.

Input($c$)
Compute $c^{(p-1)/2}$. If it is NOT 1 then output **There is no square root!**. If it is 1 then goto next step
Compute $a = c^{(p+1)/4}$.
Output $a$ and $p - a$. These are the two square roots.

**Note:** There is a similar algorithm for $p \equiv 1 \pmod{4}$ but it is slightly more complicated.
Rabin’s Encryption Scheme

$n$ is a security parameter

1. Alice gen $p, q$ primes of length $n$. Let $N = pq$. Send $N$.
2. Encode: To send $m$, Bob sends $c = m^2 \pmod{N}$.
3. Decode: Alice can find $m$ such that $m^2 \equiv c \pmod{N}$. 
Rabin’s Encryption Scheme

$n$ is a security parameter

1. Alice gen $p, q$ primes of length $n$. Let $N = pq$. Send $N$.
2. Encode: To send $m$, Bob sends $c = m^2 \pmod{N}$.
3. Decode: Alice can find $m$ such that $m^2 \equiv c \pmod{N}$. OH!
   There will be two or four of them! What to do? Later.
Rabin’s Encryption Scheme

\( n \) is a security parameter

1. Alice gen \( p, q \) primes of length \( n \). Let \( N = pq \). Send \( N \).
2. Encode: To send \( m \), Bob sends \( c = m^2 \pmod{N} \).
3. Decode: Alice can find \( m \) such that \( m^2 \equiv c \pmod{N} \). OH!
   There will be two or four of them! What to do? Later.

**BIG PRO:** Factoring Hard is hardness assumption.

**CON:** Alice has to figure out which of the sqrts is correct message.
A Theorem from Number Theory

**Definition:** A *Blum Int* is product of two primes $\equiv 3 \pmod{4}$.

**Example:** $21 = 3 \times 7$.

**Notation:** $SQ_N$ is the set of squares mod $N$. (Often called $QR_N$.)

**Example:** If $N = 21$ then $SQ_N = \{1, 4, 7, 9, 15, 16, 18\}$.

**Theorem:** Assume $N$ is a Blum Integer. Let $m \in SQ_N$. Then of the two or four sqrts of $m$, only one is itself in $SQ_N$.

**Proof:** Omitted. Note: (1) not that hard, and (2) in Katz book.

We use Theorem to modify Rabin Encryption.
Rabin’s Encryption Scheme 2.0

Also called The Blum-Williams Variant of Rabin

$n$ is a security parameter.

1. Alice gen $p, q$ primes of length $n$ such that $p, q \equiv 3 \pmod{4}$. Let $N = pq$. Send $N$.

2. **Encode:** To send $m$, Bob sends $c \equiv m^2 \pmod{N}$. Only send $m$’s in $SQ_N$.

3. **Decode:** Alice can find 2 or 4 $m$ such that $m^2 \equiv c \pmod{N}$. Take the $m \in SQ_N$.

**CON:** Messages have to be in $SQ_N$.

**History:** Had timing been different Rabin Enc would be used.
Goldwasser-Micali Encryption

$n$ is a security parameter. Will only send ONE bit. Bummer!

1. Alice gen $p, q$ primes of length $n$, and $z \in NSQ_N$. Computes $N = pq$. Send $(N, z)$.

2. Encode: To send $m \in \{0, 1\}$, Bob picks random $x \in \mathbb{Z}_N$, sends $c = z^m x^2 \pmod{N}$. Note that:
   - 2.1 If $m = 0$ then $z^m x^2 = x^2 \in SQ_N$.
   - 2.2 If $m = 1$ then $z^m x^2 = zx^2 \in NSQ_N$.

3. Decode: Alice determines if $c \in SQ$ or not. If YES then $m = 0$. If NO then $m = 1$.

**BIG PRO:** Hardness assumption natural: $SQ_N$ hard.

**BIG CON:** Messages have to be 1-bit long.

**TIME:** For one bit you need $4 \log N$ steps.
Blum-Goldwasser Enc. $n$ Sec Param, $L$ length of msg

1. Alice: $p, q$ primes len $n$, $p, q \equiv 3 \pmod{4}$. $N = pq$. Send $N$.

2. Encode: Bob sends $m \in \{0, 1\}^L$: picks random $r \in \mathbb{Z}_N$
   
   $x_1 = r^2 \pmod{N}$ \quad $b_1 = \text{LSB}(x_1)$.
   
   $x_2 = x_1^2 \pmod{N}$ \quad $b_2 = \text{LSB}(x_2)$.

   \vdots

   $x_L = x_{L-1}^2 \pmod{N}$ \quad $b_L = \text{LSB}(x_L)$.

   Send $c = (\langle m_1 \oplus b_1, \ldots, m_L \oplus b_L \rangle, x_L)$.

3. Decode: Alice: From $x_L$ Alice can compute $x_{L-1}, \ldots, x_1$ by\sqrt{\text{can do since Alice has } p, q}$. Then can compute $b_1, \ldots, b_L$ and hence $m_1, \ldots, m_L$.

BIG PRO: Hardness assumption is BBS psuedo-random.

TIME: For $L$ bits need $(L + 3) \log N$ steps. Better than Goldwasser-Micali.
LWE-KE

1. LWE-KE is a protocol for Key Exchange that does not rely on Number Theory Hardness Assumption
2. There is also a LWE-RSA.
3. These might be useful if Factoring and Discrete Log can be done by a quantum computer.
4. My presentation of it was not quite right.
5. The literature on these is not quite right either.
Good Luck on the Exam!

Good Luck on the Exam!