1. (5 points) What is your name? Write it clearly. Staple the HW.

2. (60 points) Let

\[ IS = \{ (G, k) : G \text{ has an independent set of size } \geq k \} \]

(A set of vertices, \( U \), is an Independent Set if no vertex in \( U \) has an edge to any other vertex in \( U \).

(a) Describe, in English — with pictures and an example — how you would, GIVEN a Boolean Formula \( \phi(x_1, \ldots, x_n) \) produce a graph \( G \) and a number \( k \) such that:

\[ \phi \in SAT \ \text{iff} \ (G,k) \in IS \]

(b) Write pseudocode for the procedure that takes a Boolean Formula \( \phi \) and produces \( (G,k) \), as described above.

(c) Apply your procedure to the Boolean Formula:

\[ (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2) \land x_3 \]

(d) If \( \phi \) has \( L \) clauses and each clause has three variables, then how many vertices are in \( G \) and what is \( k \)?

(e) Assume that 3-SAT is NP-complete. (See the next problem for definition of 3-SAT.) Find a function \( f \) such that:

\[ ISF = \{ G : G \text{ has } n \text{ vertices and an independent set of size } \geq f(n) \} \]

is NP-complete.

SOLUTION TO PROBLEM TWO

Suppose we have an instance of a boolean formula \( C_1 \land C_2 \land \ldots \), where each \( C_i \) is the disjunction of 3 variables. Label the variables \( x_1, \ldots x_n \) and their negations \( \neg x_1, \ldots \). Create a graph \( G \) as follows:
For each variable in each clause, create a node, labeled with the variable. For each clause, add an edge between the three nodes corresponding to the variables from that clause. Finally, for all $i$, add an edge between every pair of nodes labeled $x_i$ and the other labeled $\neg x_i$.

The conversion of 2(c) is below:
3. (25 points)

- Let \( \text{SAT} \) be the following problem: Given a Boolean formula, in the following form:

\[
C_1 \land \cdots \land C_L
\]

where each \( C_i \) is a disjunction (\( \lor \)) of literals, is the formula satisfiable?

- Let \( k\text{-SAT} \) be the following problem: Given a Boolean formula, in the following form:

\[
C_1 \land \cdots \land C_L
\]

where each \( C_i \) is a disjunction (\( \lor \)) of exactly \( k \) literals, is the formula satisfiable?

The above two points are definitions, NOT questions. HERE are the questions:

(a) Show that \( 2\text{-SAT} \) is in \( P \)

(b) Show that \( \text{SAT} \leq k\text{-SAT} \), for \( k \geq 3 \).

(PERMISSION: You may go to the web or elsewhere to find the answer; however, you must put it in your own words and understand your answer.)

SOLUTION TO PROBLEM THREE

- Note that any clause \((x_i \lor x_j)\) can be rewritten as the pair \((\neg x_i \implies x_j), (\neg x_j \implies x_i)\). Rewrite the original formula.

Create the (directed) implication graph, where each variable and its negation is a node, and a directed edge exists for every implication. Then, an instance is satisfiable if and only if no literal and its negation belongs to the same strongly connected component of its implication graph.

- Consider the clause \( C_i \). If it has only one literal, change it to \((L_1 \lor L_1 \lor L_1)\). If it has two literals, change it to \((L_1 \lor L_1 \lor L_2)\). If it has three literals, we are done. If it has more than three literals, then introduce new variables and replace the clause by:

\[
(L_1 \lor L_2 \lor z_1) \land (\neg z_1 \lor L_3 \lor z_2) \land (\neg z_2 \lor L_4 \lor z_3) \land \ldots
\]