QUESTION ONE (15 points each for 75 points total)
For each of the following sets, say if the set is:

FINITE (note: the empty set is countable)
COUNTABLE (that is, there is a bijection to $\mathbb{N}$)
UNCOUNTABLE

(Note that a function must map every element of its domain.)
AND PROVE YOUR ANSWER.

1. The set of INCREASING functions from $\mathbb{N}$ to SQUARES. (A function $f$ is INCREASING if $x < y$ implies $f(x) < f(y)$.)

2. The set of INCREASING functions from SQUARES to $\mathbb{N}$. (A function $f$ is INCREASING if $x < y$ implies $f(x) < f(y)$.)

3. The set of DECREASING functions from $\mathbb{N}$ to $\mathbb{N}$ (A function $f$ is DECREASING if $x < y$ implies $f(x) > f(y)$.)

4. The set of DECREASING functions from $\mathbb{N}$ to $\mathbb{Z}$ (A function $f$ is DECREASING if $x < y$ implies $f(x) > f(y)$.)

5. (For this homework, a function $f$ from $\mathbb{N}$ to $\mathbb{N}$ is kruskalian if $x < y$ implies $f(x) \geq f(y)$ (NOTE $\geq$ NOT $>$).) The set of kruskalian functions from $\mathbb{N}$ to $\mathbb{N}$.

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QUESTION TWO (65 points)

Consider the following proof that the rationals between 0 and 1 are uncountable. What is WRONG with the proof? (We ignore things like .999⋯ = 1, that is NOT the issue.)

Assume, by way of contradiction, that \( Q \cap [0, 1] \) is countable.

\[ q_1, q_2, q_3, \ldots \]

be a listing of those rationals. We write them out with all of their digits:

\[ q_1 = .q_{11}q_{12}q_{13} \cdots \]
\[ q_2 = .q_{21}q_{22}q_{23} \cdots \]
\[ q_3 = .q_{31}q_{32}q_{33} \cdots \]

\[ \vdots \]

We will create a rational between 0 and 1 that is NOT on the list.

Let a hat \( \hat{\cdot} \) over a number mean you add a 1 mod 10. So:

\( \hat{0} = 1 \)
\( \hat{8} = 9 \)
\( \hat{9} = 0 \).

The important thing is that \( \hat{b} \neq b \).

We form the rational:

\[ \hat{q}_{11}\hat{q}_{22}\hat{q}_{33} \cdots \]

This rational is NOT the \( i \)th on the list since it differs from \( q_i \) on the \( i \)th digit.

So the rationals between 0 and 1 are not countable.

WHAT IS WRONG WITH THIS PROOF?
QUESTION THREE (60 points)

In your own words and pictures describe an algorithm that will:

Given a regular expression $\alpha$, return an NFA that ACCEPTS exactly the strings that $\alpha$ GENERATES.