1. (40 points)

(a) (20 points) We are working in binary so all numbers are 0’s and 1’s. When we input 2 bits to a circuit we think of it as a NUMBER in base 2.
00=0
01=1
10=2
11=3
The output will be 3 bits, intepreted as a number in base 2.
000=0, . . . , 111=7
Write a truth table with 2 inputs and 3 outputs for the following function:
\[ f(xy) = (xy)^2 \pmod{8}. \]
So for example
\[ f(11) \] is computed by 11=3, \( 3^2 = 9 \pmod{8} = 1 \), so the output is 001.

(b) (20 points) Write a circuit for \( f \) using AND, OR, and NOT using the method shown in class (do not simplify- that would make it harder for the TA’s to grade!)

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2. (60 points) In this problem the inputs are considered bits that you add. So if the input is (1, 0, 1) it is NOT 101 in base 2 which is 5. Its just three bits, separate.

The Depth of a circuit is the max number of gates from input to output. The Size of a circuit is the total number of gates. 

ALSO- we allow for input the variables AND their negations.

In this problem we will look at different circuits for:

\[ f_n(x_1, \ldots, x_n) = x_1 + \cdots + x_n \pmod{2}. \]

We allow AND, OR and NOT gates usual; however, the AND and OR gates can take MANY inputs (as many as you want).

You can assume that \( n \) is a power of two if it makes the math easier.

(a) (30 points) Show that, for all \( n \), there is a circuit for \( f_n \) with depth 2. What is the circuit’s size?

(b) (30 points) Show that, for all \( n \), there is a circuit for \( f_n \) with size \( O(n) \) (less than some constant times \( n \)). What is the constant? What is the depth of the circuit? (HINT: First get a circuit for \( f_2(x_1, x_2) \). View this as a gate you can use. Use that

\[ f_n(x_1, \ldots, x_n) = f_{n/2}(f_2(x_1, x_2), f(x_3, x_4), \ldots, f(x_{n-1}, x_n)). \]

(c) (0 points but think about) In part 1 you got a CONSTANT DEPTH but LARGE SIZE circuit. In part 2 you got a SMALL SIZE but LOG DEPTH circuit. Is there a circuit for \( f_n \) which is constant depth and small size?